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Open string radiation from decaying FZZT branes

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Abstract: In this paper we continue studying the decay of unstable FZZT branes initiated in [1], [2]. The mass of tachyonic mode in this model can be chosen arbitrarily small and we use it as a perturbative parameter. In [2] a time-dependent boundary conformal field theory (BCFT) describing the decay process was studied and it was shown that in a certain sense this BCFT interpolates between two stationary BCFT's corresponding to the UV and IR fixed points of the associated RG flow. In the present work we find in the leading order vertex operators of the time-dependent BCFT. We identify the "in" and "out" vertex operators assigned to the UV and IR fixed points and compute the related Bogolyubov coefficients. We show that there is a codimension one subspace of the out-going states for which pair creation amplitudes are independent of the initial wave function of the tachyonic mode. We demonstrate that such amplitudes can be computed within the framework of first quantized open string theory via suitably defined string two-point functions. We also evaluate a three point function which we interpret as an amplitude for string triplet creation due to interaction. Some peculiarities of scattering amplitudes in the presence of tachyonic modes in the far past are discussed.

Keywords: Conformal Field Models in String Theory, Tachyon Condensation, D-branes.

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Contents

1. Introduction

Time dependent backgrounds in string theory are at present very poorly understood. As a first step one may wish to understand perturbative string amplitudes in exact time dependent backgrounds. Constructing perturbative amplitudes for such models involves making sense of functional integrals over string worldsheets that involve a field with a negative-definite metric. The last one describes a time-like direction in target space. Assuming that the ghost sector is factorized and that such integrals are defined by means of a suitable Wick rotation or otherwise, the matter part of an exact background is described by some non unitary two-dimensional conformal field theory (CFT). One can then consider vertex operators corresponding to infinitesimal deformations of the CFT at hand and define string amplitudes in the usual way by integrating the CFT correlators over the moduli space of punctured Riemann surfaces.

There is next a question of physical interpretation for such amplitudes. Clearly one should seek an S-matrix type interpretation. It seems natural to us to try following in this task as closely as possible the analogous field theoretical constructions. It is not our goal in this paper to put forward a general string-theoretic scheme for scattering in time-dependent backgrounds. Rather we will study in detail one particular model, which is well-controlled analytically, and for which we will be able to extend the main field theoretical constructions such as "in" and "out" physical states, Klein-Gordon type conserved inner product, Bogolyubov coefficients and particle creation amplitudes. For this model we will establish a relation between a string theoretic two point function and a tachyon pair creation amplitude. We will also compute a string three point function and conjecture its interpretation in terms of particle creation amplitudes.

As our considerations will essentially go in parallel with the field theoretical set up it will be instructive to recount it first. This material is fairly standard so we will be brief (see [14, 15] for a comprehensive discussion). This will be followed by a short discussion of how much of this set up can be brought over into string theory in a straightforward way and what are the problems related to the rest of the machinery. A disinterested reader may wish to go directly to the next section where the main body of the paper starts.

Consider for definiteness a scalar field $\phi(x)$ with a cubic interaction in a non-stationary spacetime. To define an S-matrix one typically restricts oneself to globally hyperbolic spacetimes and assumes that the interaction is switched off adiabatically for $t \to \pm\infty$. The solution to the equation of motion

$$[\Box_x + m^2 + \xi R_x]\phi(x) + \lambda\phi^2(x) = 0 \tag{1.1}$$

is then assumed to become asymptotically free

$$\lim_{t\to-\infty} \phi(x) = \phi_{\rm in}(x)\,, \tag{1.2}$$
$$\lim_{t\to+\infty} \phi(x) = \phi_{\rm out}(x) \tag{1.3}$$

where

$$[\Box_x + m^2 + \xi R_x]\phi_{in(out)}(x) = 0\,. \tag{1.4}$$

To construct the "in" and "out" Fock spaces one needs to define which solutions of the free asymptotic equations annihilate the vacuum. In certain spacetimes there are natural definitions of positive frequency asymptotic solutions which provide a natural definition of the "in" and "out" vacua. For example if the spacetime at hand is asymptotically stationary both in the far past and future, i.e. has asymptotic time-like Killing vectors for $t \to \pm\infty$, it is natural to define positive frequency solutions as appropriate eigenstates of those Killing vectors.

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Given a definition of positive frequency modes the construction of the "in" and "out" Fock spaces goes as follows. A conserved scalar product on the space of solutions to the free equation (1.4) is defined as

$$(\phi_1,\phi_2) = -i\int_\Sigma (\phi_1\partial_\mu\phi_2^* - \phi_2^*\partial_\mu\phi_1)\sqrt{-g}\,d\Sigma^\mu \tag{1.5}$$

where Σ^μ is the future directed surface element to a Cauchy surface Σ. Consider two complete sets of solutions u_p, u_p^* and v_q, v_q^* satisfying

$$(u_p,u_{p'}) = \delta_{p,p'}\,,\quad (u_p^*,u_{p'}^*) = -\delta_{p,p'}\,,\quad (u_p,u_{p'}^*) = 0 \tag{1.6}$$

and similarly for v_q. In addition we assume that the modes u_p are purely positive frequency as $t\to-\infty$ and the modes v_q are purely positive frequency as $t\to+\infty$. We can expand the "in" and "out" fields as

$$\begin{aligned}\phi_{\rm in}(x) &= \sum_p [a_p^{\rm in}u_p(x) + a_p^{\rm in\dagger}u_p^*(x)]\,,\\ \phi_{\rm out}(x) &= \sum_q [a_q^{\rm out}v_q(x) + a_q^{\rm out\dagger}v_q^*(x)]\,.\end{aligned} \tag{1.7}$$

The in and out Fock space vacua are defined by

$$a_p^{\rm in}|0\rangle_{\rm in} = 0\,,\qquad a_q^{\rm out}|0\rangle_{\rm out} = 0 \tag{1.8}$$

and the multiparticle states as

$$\begin{aligned}|p_1\dots p_n\rangle_{\rm in} &= \prod_{i=1}^n a_{p_i}^{\rm in\dagger}|0\rangle_{\rm in}\,,\\ |q_1\dots q_m\rangle_{\rm out} &= \prod_{i=1}^m a_{q_i}^{\rm out\dagger}|0\rangle_{\rm out}\,.\end{aligned} \tag{1.9}$$

The S-matrix elements are then defined as overlaps

$$_{\rm out}\langle q_1\dots q_m|p_1\dots p_n\rangle_{\rm in}\,. \tag{1.10}$$

In particular the amplitudes of the form

$$_{\rm out}\langle q_1\dots q_m|0\rangle_{\rm in} \tag{1.11}$$

are the particle creation amplitudes.

Since the sets of modes u_p, u_p^* and v_q, v_q^* are each complete they are related by a Bogolyubov transformation:

$$\begin{aligned}u_p &= \sum_q (\alpha_{p,q}v_q + \beta_{p,q}v_q^*)\,,\\ v_q &= \sum_p (\alpha_{p,q}^*u_p - \beta_{p,q}u_p^*)\,.\end{aligned} \tag{1.12}$$

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The orthogonality conditions (1.6) imply a number of relations between the Bogolyubov coefficients $\alpha_{p,q}$, $\beta_{p,q}$:

$$\begin{aligned} &\sum_q(\alpha_{p_1,q}\alpha^*_{p_2,q}-\beta_{p_1,q}\beta^*_{p_2,q})=\delta_{p_1,p_2}\,, && \sum_q(\alpha_{p_1,q}\beta_{p_2,q}-\beta_{p_1,q}\alpha_{p_2,q})=0\,,\\ &\sum_p(\alpha^*_{p,q_1}\alpha_{p,q_2}-\beta_{p,q_1}\beta^*_{p,q_2})=\delta_{q_1,q_2}\,, && \sum_p(\alpha^*_{p,q_1}\beta_{p,q_2}-\alpha^*_{p,q_2}\beta_{p,q_1})=0\,. \end{aligned} \tag{1.13}$$

One can show that in the case of non-interacting theory ($\lambda=0$) all of the S-matrix amplitudes (1.10) are expressible via the Bogolyubov coefficients. Only even numbers of particles can be created in this case. In particular pair creation amplitudes can be expressed as

$$_{\rm out}\langle q_1q_2|0\rangle_{\rm in}={}_{\rm out}\langle 0|0\rangle_{\rm in}\sum_p\beta^*_{p,q_1}(\alpha^{-1})^*_{q_2,p}\,. \tag{1.14}$$

Note that relations (1.13) imply that the operator $\alpha_{p,q}$ has a bounded inverse $(\alpha^{-1})_{q_2,p}$ [19].

In the interacting case one can define a modified set of Feynman rules and reduction formulas [14, 16]. In those rules one essentially separates the interaction effects which are taken care of by a suitably defined S-matrix operator and the effects due to the explicit time-dependence which are encoded in Bogolyubov's coefficients. Interaction causes additional particle creation. Thus in ϕ^3 theory there is a tree level triple creation process (see [17, 16] for some explicit computations).

Let us remark that although in the above the time dependence was coming from a time-dependent space-time metric, most of the discussion generalizes to other instances of time dependence. For example one can consider an interacting scalar field in flat spacetime coupled to an external time-dependent potential $V(x,t)$:

$$[\Box_x+m^2+V(x,t)]\phi(x)+\lambda\phi^2(x)=0\,. \tag{1.15}$$

All one needs in order to extend the above discussion to this case is some definition of positive frequency modes. For example in a case when the potential vanishes (or goes to a constant) at $t\to\pm\infty$ the definition is obvious. Note that the inner product defined in (1.5) is also conserved on solutions to (1.15).

We now turn to string theory. What follows contains some speculations concerning the structure of general formalism of perturbative string theory in time-dependent backgrounds. While supported by known examples these speculations should be taken as such. We first remark that natural analogues of wave functions are string physical states whose matter part we denote by $|V\rangle$ which are annihilated by the positive modes of Virasoro algebra: $L_n|V\rangle=\bar{L}_n|V\rangle=0$, $n>0$ and satisfy the mass shell condition

$$(L_0+\bar{L}_0)|V\rangle=2|V\rangle\,. \tag{1.16}$$

The last one is the direct analogue of the free wave equation (1.4). The 2 in the right hand side of (1.16) should be changed to 1 for the case of open strings. In the simplest situation the zero mode t of the time-like field on the worldsheet provides us with a macroscopic time and we can consider the $t\to\pm\infty$ asymptotic regions of the target space as the regions where we may be able to set up the "in" and "out" scattering states. One next would want to specify positive frequency modes in the asymptotic regions. Like in the case of quantum field theory [14, 15] such definitions depend on the particulars of the physical problem at hand. One can imagine asymptotic time-like Killing vectors to be replaced by worldsheet currents $J^{\alpha}_{\pm}(z,\bar{z})$ which are conserved in the asymptotic $t\to\pm\infty$ regions of the string Hilbert space. That is $\partial_\alpha J^{\alpha}_{\pm}\sim 0$ for $t\to\pm\infty$. In this case a basis for incoming positive frequency states can be defined in terms of on-shell states which are eigenvectors of the asymptotic charge

$$\Omega_- = i\lim_{t\to-\infty}\int dz^\alpha (J_-)_\alpha$$

of the eigenvalue ω_- with $\omega_-\geq 0$. Analogously one defines positive frequency out states as eigenvectors of positive eigenvalue for the asymptotic charge Ω_+ set up in the $t\to+\infty$ region. If V is a vertex operator creating a positive frequency state $|V\rangle$ then its Hermitean conjugate $V^\dagger$ creates a negative frequency state denoted $|V\rangle^*$.

As in the case of field theory we can specify analogs of one-particle "in" and "out" states using a conserved inner product. In general such an inner product (as well as the second quantization symplectic form) can be derived from a string field theory kinetic term. The inner product takes a particularly simple form when ghosts are factorized and the matter CFT operator L_0 has the form

$$L_0 = \partial_t^2 + \tilde{L}_0$$

where $i\partial_t$ is the time-like field zero mode momentum operator and $\tilde{L}_0$ is assumed to be unitary with respect to the BPZ inner product $\langle\ldots\rangle_{\rm BPZ}$ in the CFT state space [3]. In this situation one can define a conserved hermitean inner product on the space of solutions to the on-shell condition (1.16) as

$$(V_1,V_2)\equiv\frac{i}{2}[\langle V_1|\partial_t V_2\rangle_{\rm BPZ}-\langle V_2|\partial_t V_1\rangle_{\rm BPZ}]\,. \tag{1.17}$$

We can now pick bases of positive frequency states $|P\rangle_{\rm in}$, $|Q\rangle_{\rm out}$ that together with the conjugate states $|P\rangle^*_{\rm in}$, $|Q\rangle^*_{\rm out}$ satisfy conditions similar to (1.6) with respect to the inner product (1.17). Here P and Q are complete sets of asymptotic quantum numbers labeling the positive frequency "in" and "out" states respectively. The negative frequency states $|P\rangle^*_{\rm in}$, $|Q\rangle^*_{\rm out}$ can be interpreted as incoming or respectively outgoing string fundamental excitations or particles. Let us denote by $V^{\rm in}_P$ and $V^{\rm out}_Q$ the world sheet vertex operators corresponding to the states $|P\rangle_{\rm in}$, $|Q\rangle_{\rm out}$. Assuming both bases are complete these operators are related by Bogolyubov transformations

$$\begin{aligned} V^{\rm in}_P &= \sum_Q(\alpha_{P,Q}V^{\rm out}_Q+\beta_{P,Q}V^{\rm out\dagger}_Q)\,,\\ V^{\rm out}_Q &= \sum_P(\alpha^*_{P,Q}V^{\rm in}_P-\beta_{P,Q}V^{\rm in\dagger}_P)\,. \end{aligned} \tag{1.18}$$

Define operators

$$\mathcal{V}^{\rm in}_Q=\sum_P(\alpha^{-1})_{Q,P}V^{\rm in}_P\,.$$

These operators are pure positive frequency in the far past and in the far future their positive frequency part is $V_Q^{\rm out}$. It seems natural to us to conjecture that the string theoretic two-point function of such operators gives a normalized pair creation amplitude

$$\frac{1}{2}\langle \mathcal{V}_{Q_1}^{\rm in}\mathcal{V}_{Q_2}^{\rm in}\rangle_{\rm str} = \sum_P \beta_{P,Q_1}(\alpha^{-1})_{Q_2,P} = \frac{{}_{\rm in}\langle 0|Q_1Q_2\rangle_{\rm out}}{{}_{\rm in}\langle 0|0\rangle_{\rm out}}\,. \tag{1.19}$$

A similar conjecture was put forward in [8] regarding certain CFT two-point functions and pair creation rates. While it may work for a certain type of models such as time-like Liouville Theory considered in [8], it seems to us that in general, when the time-like part of the CFT and the spatial part are mixed (as in the model considered in this paper) one should consider the string two-point function. The main difficulty with this proposal is technical. There is no general prescription for computing a string-theoretic two-point function. A formal expression for a two-point function in a noncompact target space contains an infinity coming from integrating over target space zero modes and a vanishing factor arising from the division by the infinite volume of the group of worldsheet modular transformations fixing two points. The cancellation of the two infinities is relatively well understood only for noncritical strings [10, 9, 12, 13].

String theory is an interacting theory. There should be prescriptions to compute multiparticle S-matrix amplitudes of the type (1.10) that take into account string interactions. We conjecture that whenever string n-point functions of operators $\mathcal{V}_Q^{\rm in}$ can be defined they give perturbative contributions (at each genus) to the n-particle creation amplitudes

$$\langle \mathcal{V}_{Q_1}^{\rm in}\mathcal{V}_{Q_2}^{\rm in}\dots\mathcal{V}_{Q_n}^{\rm in}\rangle_{\rm str} = C_n\frac{{}_{\rm in}\langle 0|Q_1Q_2\dots Q_n\rangle_{\rm out}}{{}_{\rm in}\langle 0|0\rangle_{\rm out}}$$

where C_n is a numerical normalization factor.

In the above discussion there were no specific points referring to closed strings so if correct the same scheme should apply also to open string time-dependent backgrounds. Also very mild assumptions were made on the nature of the time-dependence. In this paper we consider a particular model describing a time dependent process of open string tachyon condensation in two-dimensional string theory. We will find that the presence of tachyon instability in the initial system brings about certain additional complications into the above general scheme. Namely solutions exponentially growing in the far past are needed for completeness of the out scattering states. We suspect this to be a generic situation for decays of unstable backgrounds. This results on the one hand in an additional ambiguity in defining the initial state of the system and on the other hand, from the CFT perspective, in the need to define correlation functions for exponentially blowing up operators. The formalism of first quantized string theory per se does not contain a prescription to compute such correlation functions. Nevertheless we will show that for a large class of vertex operators (for a codimension one subspace in the total space of solutions) the correlation functions do not depend on these additional ambiguities. We will demonstrate that for these states the appropriately defined string two point functions give pair creation amplitudes. We will also compute a string three point amplitude and conjecture its interpretation in terms of particle triplet creation rate. A more detailed discussion of our results is given in the final section of the paper.

The main body of the paper is organized as follows. In section 2 we introduce the model we study and give a review of main results obtained in [1, 2]. In section 3 we explain our approach to constructing vertex operators in our time-dependent model and compute explicitly vertex operators asymptoting to plane waves in the infinite past. In section 4 we discuss the conserved inner product and define normalized "in" and "out" states. In section 5 a basis of vertex operators asymptoting to positive frequency out states are constructed and the exponential blow up of such solutions in the infinite past is demonstrated. In section 6 we analyze the Bogolyubov transformation relating the "in" and "out" vertex operators. In section 7 we develop the secondary quantization of the model. We construct a family of physically natural initial states in the oscillators state space. In section 8 a string theoretic two point function is computed in a certain regularization scheme. It is shown that for a large (codimension one) class of out states it gives a pair production rate. In section 9 a string three point function is computed. In the final section we discuss our results and point at some open questions. The appendix contains some technical details of the computations.

2. The model

In this section we explain the particulars of the model and review the main results obtained in [1, 2]. Our model is constructed in the framework of $c = 1$ noncritical string theory (see e.g. [9] for a review). The worldsheet CFT is a product of a free timelike boson X_0 and the $c = 25$ Liouville theory. The action for X_0 is

$$S_{X_0} = -\frac{1}{4\pi}\int d^2x\,(\nabla X_0)^2 \tag{2.1}$$

where the sign in front of the action means that X_0 is timelike. The Liouville theory is a conformal field theory of an interacting noncompact boson ϕ with the action functional

$$S_L = \frac{1}{4\pi}\int d^2x[(\nabla\phi)^2 + 4\pi\mu e^{2b\phi}] \tag{2.2}$$

and the background charge $Q = b + 1/b$ is introduced via the the asymptotic at spatial infinity $\phi(x) \sim -Q\log x^2$. This theory is by now well understood and we refer the reader to [4] for a review of essential results. From now on we set $b = 1$ that corresponds to the central charge $c = 25$. At the level of sigma model description the $c = 25$ Liouville theory is characterized by a flat 2D metric and the following backgrounds for the dilaton and tachyon fields Φ and T:

$$\Phi(\phi, X_0) = \phi\,, \qquad T(\phi, X_0) = \mu e^{2\phi}\,. \tag{2.3}$$

The linear dilaton profile implies that the string coupling goes to zero in the $\phi \to -\infty$ region and blows up in the $\phi \to +\infty$ region. The strings however are essentially confined to the weakly coupled region by the tachyon potential. String scattering in and out states are naturally set up in the $\phi \to -\infty$ asymptotic region.

We are further interested in the open string sector of this theory which is introduced via Neumann type conformal boundary conditions. The corresponding D1-branes are referred to in the literature as FZZT branes after the authors of [5, 6]. At the semiclassical level these boundary conditions are defined by adding to the bulk theory put on an upper half plane $\{(x,\tau)|x \in \mathbb{R}, \tau \geq 0\}$ a boundary action

$$S_\partial = \mu_B \int_{\mathbb{R}} dx\, e^{\phi} \tag{2.4}$$

which results in the boundary conditions

$$i(\partial - \bar{\partial})\phi = 2\pi\mu_B e^{\phi}\,. \tag{2.5}$$

By solving the corresponding boundary conformal field theory (BCFT) exactly it was found that at the quantum level for each value of the boundary coupling μ_B there are countably many physically distinct boundary conditions [5−7, 1]. The quantum boundary conditions are parameterized by a parameter δ related to μ_B via[1]

$$\cos[\pi(1+\delta)] = \frac{\mu_B}{\sqrt{\mu}}\,. \tag{2.6}$$

In particular the spectrum of boundary operators depends on the value of δ. Before we discuss the latter a note on the choice of worldsheet is in order. For the most part of the paper our computations will be done on a disc $\mathbb{D} = \{(r,\sigma)|0 \leq r \leq 1, 0 \leq \sigma \leq 2\pi\}$. This concerns in particular the computations of the two and three point functions in sections 8 and 9. To use the state-operator correspondence in a BCFT we invoke a Hamiltonian quantization on a strip $\mathbb{S} = \{(\sigma,\tau)|0 \leq \sigma \leq \pi\}$ with the boundary condition specified by δ imposed on both edges of the strip. The corresponding Hilbert space has the form

$$\mathcal{H}^B_{\delta\delta} = \int_{\mathbb{R}_+} dP\, \mathcal{V}_P \oplus \begin{cases} \emptyset & \text{for} \quad \delta < 0\,, \\ \mathcal{V}_\vartheta & \text{for} \quad \delta > 0\,, \end{cases} \tag{2.7}$$

where $\vartheta = i\delta$ and $\mathcal{V}_Q$ is the irreducible unitary representation of the Virasoro algebra with $c = 25$ and the highest weight $\Delta_Q = 1 + Q^2$. The above representation means that for each value of δ there is a continuum of δ-function normalizable states with weights $\Delta_P = 1 + P^2\,, P \in \mathbb{R}_+$ whose boundary fields we denote $\Phi^\delta_P(x)$. In addition for $\delta > 0$ there is a single discrete normalizable state whose vertex operator we denote by $\Phi^\delta_\vartheta(x)$. The conformal weight of this operator is $\Delta_\vartheta = 1 - \delta^2$. In many manipulations it can be treated on equal footing with the operators Φ^δ_P if one regards it as an operator Φ^δ_P with $P = \vartheta = i\delta$. We choose normalizations of our operators as in [2] to be given in more detail shortly.

For the X_0 field we consider the Neumann boundary condition. The corresponding Virasoro primary states are denoted $|\omega\rangle_{X_0}$:

$$L_0^{\text{open}}|\omega\rangle_{X_0} = -\omega^2|\omega\rangle_{X_0} \quad L_n^{\text{open}}|\omega\rangle_{X_0} = 0\,,\ \ n > 1\,. \tag{2.8}$$

[1]The parameter δ is related to the parameter σ from [7, 1] as $2\sigma = 1-\delta$ and to the parameter s from [5] as $s = i(1+\delta)$.

We introduce zero modes of the fields as

$$t = \int_0^\pi d\sigma\, X_0(\sigma,0) \quad \phi_0^{\rm op} = \int_0^\pi d\sigma\, \phi(\sigma,0)\,. \tag{2.9}$$

This allows us to define the wave functions for the highest weight states $|P\rangle \otimes |\omega\rangle_{X_0}$ as

$$\Psi_P(\phi_0^{\rm op}, t) = \langle \phi_0^{\rm op}|P\rangle \cdot \langle t|\omega\rangle_{X_0} = \langle \phi_0^{\rm op}|P\rangle \cdot e^{i\omega t} \tag{2.10}$$

and similarly with P replaced by ϑ. In the weak coupling region $\phi_0^{\rm op} \to -\infty$ the Liouville parts of wavefunctions behave as

$$\Psi_P(\phi_0^{\rm op}) \sim C_\delta(P) e^{iP\phi_0^{\rm op}} + C_\delta(-P) e^{-iP\phi_0^{\rm op}} \tag{2.11}$$

where $C_\delta(P)$ is a certain normalization factor. One can show (see e.g. [2] appendix B.1) that $C_\delta(\vartheta) = 0$ and thus the wavefunction $\Psi_\vartheta(\phi_0^{\rm op})$ decays exponentially in the asymptotic region that is characteristic of a bound state.

Since Δ_ϑ is less than one the corresponding open string state is tachyonic. The string spectrum contains an unstable excitation with the vertex operator $\Phi_\vartheta^\delta e^{\delta X_0}$ whose wavefunction increases exponentially with t. We thus have a system with a localized open string tachyon whose mass δ can be chosen to be arbitrarily small. From the target space perspective the smallness of δ means that the tachyon condensation process is slow rolling. This process can be described by deforming the (Liouville) $\times$ X_0 BCFT adding to it a boundary interaction term generated by the tachyon vertex operator

$$S_\lambda = \lambda \int_{\mathbb{R}} dx\, [\Phi_\vartheta^\delta e^{\delta X_0}](x)\,. \tag{2.12}$$

Looking at the operator product expansions of the multiple products of the tachyon operator with itself it is easy to see that no divergences arise when treating the interaction term perturbatively. The deformed theory is therefore conformal. The smallness of the parameter δ can then be used by employing the RG resummation technique to construct an effective Lagrangian that gives the boundary state to the leading order in δ [2].

For future reference we give here the details of the spectrum of boundary operators of the FZZT branes in the $\delta \to 0$ limit. The two point functions in the normalizations of [2] are

$$\langle \Phi_P^\delta(x_1)\Phi_{P'}^\delta(x_2)\rangle = |x_1 - x_2|^{-2\Delta_P} C_\delta(P)C_\delta(-P)\delta(P-P')\,, \tag{2.13}$$

$$\langle \Phi_\vartheta^\delta(x_1)\Phi_\vartheta^\delta(x_2)\rangle = |x_1 - x_2|^{-2\Delta_\vartheta} d_\delta \tag{2.14}$$

where the factors $C_\delta(P)$ and d_δ in the leading order are

$$C_\delta(P) \sim \mu_r^{1/2} \frac{\pi(\delta + iP)}{iP}\,, \quad d_\delta \sim \frac{\pi\mu_r}{\delta}\,. \tag{2.15}$$

Here and elsewhere when taking the $\delta \to 0$ asymptotics appropriate for conformal perturbation theory one should assume that the Liouville momenta P are all of the order δ (see [2] for a detailed explanation). We will be often using the rescaled variables

$$p = P/\delta \quad q = Q/\delta\,, \text{ etc.} \tag{2.16}$$

which are of the order δ^0.

The operator product expansions have the form

$$\Phi^\delta_{P_2}(x_2)\Phi^\delta_{P_1}(x_1) = \int_0^\infty dP_3\, F^{P_3}_{P_2P_1}|x_2-x_1|^{\Delta_{P_3}-\Delta_{P_2}-\Delta_{P_1}}\Phi^\delta_{P_3}(x_1) + \tag{2.17}$$
$$+f^{\vartheta}_{P_2P_1}|x_2-x_1|^{\Delta_\vartheta-\Delta_{P_2}-\Delta_{P_1}}\Phi^\delta_\vartheta(x_1) + \text{ descendants}$$

where P_2 and P_1 can also assume the value ϑ. The asymptotic formulas for the OPE coefficients are

$$F^{P_3}_{P_2P_1} \sim \frac{2P_3^2}{\pi(\delta^2+P_3^2)}\,, \tag{2.18}$$
$$f^{\vartheta}_{P_2P_1} = -2\pi i\mathrm{Res}_{P3=i\delta}F^{P_3}_{P_2P_1} \sim 2\delta \tag{2.19}$$

where P_1 and P_2 can take the value ϑ.

The three point functions to the leading order in δ all take the same value

$$\langle\Phi^\delta_{P_1}(x_1)\Phi^\delta_{P_2}(x_2)\Phi^\delta_{P_3}(x_3)\rangle \sim 2\pi|x_1-x_2|^{\Delta_{12}}|x_2-x_3|^{\Delta_{23}}|x_3-x_1|^{\Delta_{13}} \tag{2.20}$$

where Δ_{ij} are standard combinations of conformal dimensions. In the last formula any of P_i's can take the value ϑ.

The leading order contributions from interaction (2.12) to correlation functions come from short distances. Hence, although the theory is finite, one can use the RG resummation technique to construct the effective Lagrangian [2]. Introducing a short distance cutoff ϵ we write down a renormalized boundary action that includes all operators near-marginal in the $\delta \ll 1$ limit which are generated via short distance expansions

$$S^{\rm ren}_\lambda = \sum_{n=1}^{\infty}\int_{\mathbb{R}} dx\left(U_n\epsilon^{(n^2-1)\delta^2}[e^{n\delta X_0}\Phi^\delta_\vartheta](x) + \int_0^\infty dP\,\lambda_n(P)\epsilon^{n^2\delta^2+P^2}[e^{n\delta X_0}\Phi^\delta_P](x)\right). \tag{2.21}$$

The RG equations arise as conditions for ϵ-independence of the correlation functions. These equations supplemented by the appropriate conditions fixing the bare couplings can be solved explicitly [2]. The results are most elegantly summarized in terms of generating functions

$$\lambda(q,t) = \sum_{n=1}^{\infty}\lambda_n(q\delta)e^{n\delta t}\,, \quad U(t) = \sum_{n=1}^{\infty}U_n e^{n\delta t} \tag{2.22}$$

where the parameter t can be identified with the zero mode of the X_0 field. The function $U(t)$ can be expressed via elementary functions and the function $\lambda(q,t)$ via the hypergeometric function ${}_2F_1$. The explicit expressions can be found in [2] and will not be used in this paper.

A combination of the generating functions that will be important later is

$$W(t) = U(t) + \delta\int_0^\infty dq\,\lambda(q,t)\,. \tag{2.23}$$

It has a simple explicit expression

$$W(t) = \delta\left(\frac{\nu e^{\delta t}}{1+\nu e^{\delta t}}\right) \tag{2.24}$$

where $\nu = \lambda/\delta$ - the rescaled bare coupling λ from (2.12).

Although the generating functions (2.22) ab initio had a radius of convergence bounded by $t \ll \delta^{-1}$ they have a natural analytic continuation for all values[2] of t. In particular one can find the $t \to \infty$ asymptotic which can be interpreted as the far future of the FZZT brane decay process. The generating function $U(t)$ tends to a constant $u_* = 2\delta$ while the couplings of the continuous operators have asymptotics

$$\lambda(q,t) \to -\frac{2}{\pi(1+q^2)} + \frac{q}{\sinh(\pi q)}\left(\frac{e^{iq\delta t}\nu^{iq}}{1+iq} + \frac{e^{-iq\delta t}\nu^{-iq}}{1-iq}\right). \tag{2.25}$$

It was further shown that the constant (t-independent) parts of the above asymptotics are described by the boundary condition characterized by the parameter $-\delta$ while the oscillatory piece in (2.25) was interpreted as radiation. The conclusion of [2] was that the $0 < \delta \ll 1$ FZZT brane decays into the $\delta_* = -\delta$ brane leaving behind a radiation travelling towards $\phi_0^{\rm op} = -\infty$.

It is one of the purposes of the present paper to derive the radiation produced in the decay process from first principles.

3. Time-dependent vertex operators

3.1 First order conformal deformation equations

We would like to find marginal operators of the time-dependent BCFT (2.12). In general marginal operators of a given (B)CFT can be identified with its infinitesimal deformations. Consider an infinitesimal perturbation of the boundary theory (2.4), (2.12) by a term

$$\xi_0 \int_{\mathbb{R}} dx\, [e^{iPX_0}\Phi^{\delta}_{|P|}](x)$$

where ξ_0 is a constant (the deformation parameter) and P is any real number except zero. The $P = 0$ case needs special care and will be treated separately later. The perturbing operator $e^{iPX_0}\Phi^{\delta}_{|P|}$ is a primary of dimension 1 relative to the undeformed (FZZT) $\times\, X_0$ boundary theory. The tachyon interaction term (2.12) will result in mixing of this operator with operators $e^{(n\delta+iP)X_0}\Phi^{\delta}_{Q}$, $e^{(n\delta+iP)X_0}\Phi^{\delta}_{\vartheta}$, $n \in \mathbb{Z}_+$ that can be found via operator product expansion. Thus we are led to consider a combined renormalized action

$$S^{\rm ren}_{\rm Bd} = S^{\rm ren}_{\lambda} + \Delta S^{\rm ren}_{\mu,\eta} \tag{3.1}$$

[2]It is interesting to note in regard with the analytic continuation that it is well defined only for $\nu > 0$. For negative values of ν the analytically continued solution will hit a branch cut for sufficiently large values of t. This can be correlated with the fact that there is no perturbative fixed point for the RG flow triggered by the relevant operator $\Phi^{\delta}_{\vartheta}$ with a negative coupling constant

where $S^{\rm ren}_{\lambda}$ is given by (2.21) and

$$\Delta S^{\rm ren}_{\mu,\eta} = \int\limits_{\mathbb{R}} dx \sum_{n=0}^{\infty} \int dQ\, \mu_n(P,Q) \epsilon^{(n\delta+iP)^2+Q^2} [e^{(n\delta+iP)X_0} \Phi^{\delta}_{Q}](x) + \\ + \int\limits_{\mathbb{R}} dx \sum_{n=1}^{\infty} \eta_n(P) \epsilon^{(n\delta+iP)^2-\delta^2} [e^{(n\delta iP)X_0} \Phi^{\delta}_{\vartheta}](x)\,. \tag{3.2}$$

The last perturbation is considered only to first order in $\mu_n(P,Q)$ and $\eta_n(P)$.

The beta functions for the couplings $\mu_n(P,Q)$ and $\eta_n(P)$ can be computed via conformal perturbation theory as in [2]. It is clear from the general form of the OPE's at hand that the RG equations for the original couplings U_n and $\lambda_n(P)$ are unaffected by the presence of the new couplings[3] in $\Delta S^{\rm ren}_{\mu,\eta}$ and are given by the solution found in [2]. The additional RG equations for the new couplings read

$$\begin{aligned}
\epsilon \frac{d\mu_n(P,Q)}{d\epsilon} &= -(Q^2 + (n\delta+iP)^2)\mu_n(P,Q) - 2\sum_{l=1}^{n} \int dQ'\, F^{Q}_{\vartheta Q'} U_l \mu_{n-l}(P,Q') - \\
&\quad -2\sum_{l=1}^{n} \Big[\int\!\!\int dQ' dQ''\, F^{Q}_{Q'Q''} \lambda_l(Q'') \mu_{n-l}(P,Q') \\
&\qquad + F^{Q}_{\vartheta P} U_l \eta_{n-l}(P) + \int dQ'\, \lambda_l(Q') \eta_{n-l}(P) \Big] \\
\epsilon \frac{d\eta_n(P,Q)}{d\epsilon} &= (\delta^2 - (n\delta+iP)^2)\eta_n(P) - 2\sum_{l=1}^{n-1} f^{\vartheta}_{\vartheta\vartheta} U_l \eta_{n-l}(P) \\
&\quad - \sum_{l=1}^{n} \Big[\int dQ\, f^{\vartheta}_{\vartheta Q} \lambda_l(Q) \eta_{n-l}(P) + \int dQ\, f^{\vartheta}_{\vartheta Q} U_l \mu_{n-l}(P,Q) \\
&\qquad + \int\!\!\int dQ dQ'\, f^{\vartheta}_{QQ'} \lambda_l(Q') \mu_{n-l}(P,Q) \Big]\,.
\end{aligned} \tag{3.3}$$

The supplementary renormalization conditions are

$$\begin{aligned}
\lim_{\epsilon\to 0} \eta_n(P) \epsilon^{(n\delta+iP)^2-\delta^2} &= 0\,, \\
\lim_{\epsilon\to 0} \mu_n(P,Q) \epsilon^{(n\delta+iP)^2+Q^2} &= 0 \quad \text{for } n>0\,, \\
\lim_{\epsilon\to 0} \mu_0(P,Q) \epsilon^{Q^2-P^2} &= \xi_0 \delta(P-Q)
\end{aligned} \tag{3.4}$$

where ξ_0 is constant. The above equations with these conditions can be solved recursively and are equivalent to

$$\epsilon \frac{d\mu_n(P,Q)}{d\epsilon} = 0 = \epsilon \frac{d\eta_n(P)}{d\epsilon} \tag{3.5}$$

which means, as anticipated, that the corresponding perturbation generated by a nonvanishing ξ_0 is first order marginal.

[3] This is true only for $P \neq 0$.

An additional comment perhaps is in order on the meaning of the renormalized action (3.1). The renormalized action (2.21) is an effective action that can be used to compute the leading orders in δ of the disc partition function and one-point functions of the bulk operators of the time dependent BCFT (2.12). The renormalized action (3.1) with the constants μ_n and η_n treated to the first order can be used to compute the leading order in δ of the correlators of marginal *boundary* operators of the BCFT (2.12). Solutions to the (3.3) substituted into (3.2) thus give renormalized boundary marginal operators labeled by P:

$$\Phi_P(\nu) = \sum_{n=0}^{\infty}\int_0^{\infty} dQ\,\mu_n(P,Q)[e^{(n\delta+iP)X_0}\Phi_Q^\delta] + \sum_{n=1}^{\infty}\eta_n(P)[e^{(n\delta iP)X_0}\Phi_\vartheta^\delta] \tag{3.6}$$

where we included the coupling constant ν in the notation to signify that these are primaries of the deformed theory (2.12).

In the leading order in δ one uses the asymptotic expressions for the OPE coefficients (2.18), (2.19) in the above equations to obtain

$$\begin{aligned}(Q^2+(n\delta+iP)^2)\mu_n(P,Q) &= -2f(Q)\sum_{l=1}^{n}W_l h_{n-l}(P)\,,\\ (\delta^2-(n\delta+iP)^2)\eta_n(P) &= 4\delta\sum_{l=1}^{n}W_l h_{n-l}(P)\end{aligned} \tag{3.7}$$

where

$$h_n(P) = \eta_n(P) + \int_0^{\infty} dQ\,\mu_n(Q,P)\,, \tag{3.8}$$

$$f(Q) = \frac{2Q^2}{\pi(\delta^2+Q^2)} \tag{3.9}$$

and the coefficients W_n can be read off the generating function $W(t)$ given in (2.23), (2.24). Note that while a particular renormalization scheme was used to obtain (3.3) the leading order equations (3.7) are scheme independent.

The above equations look most compact when written in terms of generating functions

$$\begin{aligned}\mu(t,P,Q) &= \sum_{n=0}^{\infty}\mu_n(P,Q)e^{(n\delta+iP)t}\,,\\ \eta(t,P) &= \sum_{n=1}^{\infty}\eta_n(P)e^{(n\delta+iP)t}\\ h(t,P) &= \sum_{n=0}^{\infty}h_n(P)e^{(n\delta+iP)t} = \eta(t,P)+\int_0^{\infty}dQ\,\mu(t,P,Q)\,.\end{aligned} \tag{3.10}$$

Identifying as before the parameter t with the target space time (the zero mode of X_0) we can think of these generating functions as time-dependent coupling constants. Equations (3.7) read

$$\begin{aligned} -(Q^2+\partial_t^2)\mu(t,P,Q) &= 2f(Q)W(t)h(t,P)\,, \\ (\partial_t^2-\delta^2)\eta(t,P) &= -4\delta W(t)h(t,P)\,. \end{aligned} \tag{3.11}$$

Assuming that for t sufficiently small: $t \ll \delta^{-1}$ the series expansions (3.10) converge the renormalization conditions (3.4) imply the initial conditions

$$\begin{aligned} \lim_{t\to-\infty}\mu(t,P,Q) &= e^{iPt}\xi_0\delta(P-Q) \\ \lim_{t\to-\infty}\eta^{\pm}(t,P) &= 0 \end{aligned} \tag{3.12}$$

that means that the wave functions for operators (3.6) in the infinite past look like

$$\xi_0 e^{iPt}\Psi_{|P|}(\phi_0^{\rm op})\,.$$

The conditions (3.12) are to be used as initial conditions for solving the differential equations (3.11).

Note that unlike the mode equations (3.7) the continuous equations (3.11) do not carry any reference to the initial condition and thus should be regarded as more fundamental. These equations are the direct analogue of the on-shell condition (1.16). Note also their similarity to (1.15).

3.2 Solutions asymptoting to plane waves in the far past

We would like to find an explicit solution to equations (3.7) with boundary conditions (3.4) or equivalently (3.11), (3.12). We will be using both forms of equations interchangeably.

With the given boundary conditions (3.4) we can rewrite the first equation in (3.7) as

$$\mu_n(P,Q) = -\frac{2f(Q)}{(Q^2+(n\delta+iP)^2)}\sum_{l=1}^{n}W_l h_{n-l}(P) + \xi_0\delta(P-Q)\delta_{n,0}\,. \tag{3.13}$$

Introducing

$$\xi_n(P) \equiv \int_0^\infty dQ\,\mu_n(P,Q) \tag{3.14}$$

we obtain by integrating (3.13) over Q

$$(\delta(n+1)+iP)\xi_n(P) = -2\sum_{l=1}^{n}W_l h_{n-l}(P) + \delta_{n,0}\xi_0(\delta+iP)\,. \tag{3.15}$$

The second equation in (3.7) can be rewritten as

$$(\delta(n+1)+iP)(\delta(n-1)+iP)\eta_n(P) = -4\delta\sum_{l=1}^{n}W_l h_{n-l}(P) \tag{3.16}$$

Taking an appropriate linear combination of the last two equations we obtain

$$(\delta(n-1)+iP)h_n(P) = -2\sum_{l=1}^{n}W_l h_{n-l}(P) + \xi_0(-\delta+iP)\delta_{n,0} \tag{3.17}$$

that is an equation on the modes $h_n(P)$. We proceed by solving first this equation and then substituting the solution into (3.13), (3.16) and their continuous counterparts (3.11).

Switching to the generating functions we rewrite (3.17) as

$$(\partial_t - \delta)h(t,P) = -2W(t)h(t,P) + \xi_0(-\delta + iP)e^{iPt} . \tag{3.18}$$

It is straightforward to find the solution to this equation

$$h(t,P) = \xi_0\left(\frac{2W^2(t)}{\delta^2 ip(1+ip)} - \frac{2W(t)}{\delta ip} + 1\right)e^{iPt} . \tag{3.19}$$

The corresponding modes $h_n(P)$ can be plugged into (3.13), (3.16) to obtain series expansions for $\mu(t,P,Q)$ and $\eta(t,P)$ in the variable $x = \nu e^{\delta t}$. One obtains

$$\begin{aligned} \mu(t,P,Q) &= \xi_0\delta(Q-|P|) + i\frac{f(Q)}{Q}\hat{D}_p(x)[\Phi(-x,1,1+i(p-q)) - \Phi(-x,1,1-i(p-q))] , \\ \eta(t,P) &= 2\hat{D}_p(x)[\Phi(-x,1,2+ip) - \Phi(-x,1,ip)] \end{aligned} \tag{3.20}$$

where $\hat{D}_p(x)$ is a differential operator

$$\hat{D}_p(x) = x + \frac{2x^2}{ip}\frac{d}{dx} + \frac{x^3}{ip(1+ip)}\frac{d^2}{dx^2} \tag{3.21}$$

and $\Phi(z,s,v)$ stands for the Lerch phi-function (see e.g. [18] section 9.55). The last one is defined in the region $|z| < 1$ and for $v \neq 0, -1, -2, \ldots$ by a power series expansion

$$\Phi(z,s,v) = \sum_{n=0}^{\infty} \frac{z^n}{(n+v)^s} \tag{3.22}$$

and is analytically extended to the complex plane with a branch cut going over the real line from $z = 1$ to $z = +\infty$. For $s = 1$, which is the case at hand, the Lerch phi function can be expressed via the hypergeometric ${}_2F_1$ function as

$$\Phi(z,1,v) = \frac{{}_2F_1(1,v,1+v;z)}{v} .$$

For future reference we record here the identity

$$\Phi(z,1,v) = \frac{\pi}{\sin(\pi v)}(-z)^{-v} + z^{-1}\Phi(z^{-1},1,1-v) \tag{3.23}$$

which can be used to obtain the asymptotic expansion near $z = \infty$.

We can thus conclude from (3.20), (3.21) that although the perturbation series for the time-dependent couplings $\mu(t,P,Q)$ and $\eta(t,P)$ is initially set up for sufficiently large and negative values of t, more precisely for $t < -\delta^{-1}\ln\nu$, it can be extended via analytic continuation in the variable x to all values of t. In particular one can study the $t \to +\infty$ asymptotic.

Although the above representation of solutions via Lerch phi-function is important in establishing its convergence properties in practice we will find it more useful to work with their spectral representations. With the function $h(t,P)$ given explicitly in (3.19) equations (3.11) take the form of oscillator equations with a driving force. They are solved by passing to the Fourier transforms. Taking into account the initial conditions (3.12) we obtain

$$\mu(t,P,Q) = e^{iPt}\xi_0\delta(|P|-Q) + f(Q)\xi_0 \int\limits_{-\infty}^{+\infty} d\omega\, e^{-i\omega t}\frac{d_P(\omega)}{(\omega+i\epsilon)^2-Q^2}\,,$$

$$\eta(t,P) = 2\xi_0\delta \int\limits_{-\infty}^{+\infty} d\omega\, e^{-i\omega t}\frac{d_P(\omega)}{\delta^2+\omega^2} \tag{3.24}$$

where

$$d_P(\omega) = \frac{2\omega(1+i\omega/\delta)}{\delta p(1+ip)}\hat{W}(\omega+P) \tag{3.25}$$

and

$$\hat{W}(\omega) = \frac{i\nu^{-i\omega/\delta}}{2\sinh[\pi(\omega+i\epsilon)/\delta]} \tag{3.26}$$

is the Fourier transform of $W(t)$.

Note that the corresponding operators $\Phi_P(\nu)$ defined in (3.6) satisfy the following hermitean conjugation rule

$$\Phi_P(\nu)^\dagger = \Phi_{-P}(\nu)\,. \tag{3.27}$$

3.3 Zero momentum solutions

The case $P=0$ needs special care. One notices that solutions (3.24) do not have a $P\to 0$ limit unless the normalization factor $\xi_0 = \xi_0(P)$ vanishes at least as fast as P. We will see in the next section that a natural normalization factor for our solution is such that it vanishes only as $|P|^{1/2}$ that does not compensate the blow up of the spectral function $d_P(\omega)$ in the $P\to 0$ limit. The physical reason for this apparent singularity is that at $P=0$ the RG equations for the coupling constants $u(t)$, $\lambda(q,t)$ are no longer independent from the $\eta(t)$, $\mu(t,Q)$ variables.

One can find two distinct solutions at zero momentum by taking limits of suitable linear combinations of solutions (3.24). We first consider a solution $\delta(\nu)$ defined as

$$\delta(\nu) \equiv \lim_{P\to 0} P\Phi_P(\nu) \tag{3.28}$$

where we took the normalization factor ξ_0 to be identically one. The corresponding coupling constants are

$$\mu_{\delta\nu}(t,Q) = f(Q)\int\limits_{-\infty}^{+\infty} d\tilde{\omega}\, e^{-i\tilde{\omega}\delta t}\frac{d_{\delta\nu}(\tilde{\omega})}{[(\tilde{\omega}+i\epsilon)^2-q^2]}\,,$$

$$\eta_{\delta\nu}(t) = 2\delta\int\limits_{-\infty}^{+\infty} d\tilde{\omega}\, e^{-i\tilde{\omega}\delta t}\frac{d_{\delta\nu}(\tilde{\omega})}{(1+\tilde{\omega}^2)} \tag{3.29}$$

where $\tilde{\omega} = \omega/\delta$ and

$$d_{\delta\nu}(\tilde{\omega}) = 2\tilde{\omega}(1 + i\tilde{\omega})\hat{W}(\delta\tilde{\omega})\,. \tag{3.30}$$

It is straightforward to check that this solution describes the marginal operator corresponding to variation of the bare coupling constant ν. One has

$$\mu_{\delta\nu}(t,Q) = 2i\nu\frac{\partial}{\partial\nu}\lambda(Q,t)\,, \quad \eta_{\delta\nu}(t) = 2i\nu\frac{\partial}{\partial\nu}U(t) \tag{3.31}$$

where $\lambda(Q,t)$ and $U(t)$ were found in [2]. The solution $\delta(\nu)$ asymptotes to zero at $t \to -\infty$:

$$\lim_{t\to-\infty}\mu_{\delta\nu}(t,Q) = 0\,, \quad \lim_{t\to-\infty}\eta_{\delta\nu}(t) = 0\,. \tag{3.32}$$

The existence of such a solution means that one cannot fix a unique solution to equations (3.11) by fixing its leading asymptotic at $t \to -\infty$. In order to fix a solution uniquely one should also fix the first subleading terms in the series expansion in $e^{\delta t}$ for $t \ll 0$. It suffices to fix the coefficient η_1 for that purpose.

Another zero momentum solution which we denote $\Phi_0(\nu)$ corresponds to a perturbation at $t \to -\infty$ by a cosmological constant operator Φ_0^δ. It can be obtained as a limit

$$\Phi_0(\nu) \equiv \lim_{P\to 0}\frac{1}{2}(\Phi_P(\nu) + \Phi_{-P}(\nu))\,. \tag{3.33}$$

where the normalization factor ξ_0 is again chosen to be identically one. The coupling constants corresponding to (3.33) read

$$\mu_0(t,Q) = \delta(Q) + \frac{f(Q)}{\delta}\int_{-\infty}^{+\infty} d\tilde{\omega}\, e^{-i\tilde{\omega}\delta t}\frac{d_0(\tilde{\omega})}{[(\tilde{\omega}+i\epsilon)^2 - q^2]}\,, \tag{3.34}$$

$$\eta_0(t) = 2\int_{-\infty}^{+\infty} d\tilde{\omega}\, e^{-i\tilde{\omega}\delta t}\frac{d_0(\tilde{\omega})}{(1+\tilde{\omega}^2)} \tag{3.35}$$

where

$$d_0(\tilde{\omega}) = \lim_{P\to 0}\frac{1}{2}(d_P(\tilde{\omega}) + d_{-P}(\tilde{\omega})) = -i(1+\ln(\nu))d_{\delta\nu}(\tilde{\omega}) + \tilde{\omega}(\tilde{\omega}-i)\pi\frac{\nu^{-i\tilde{\omega}}\cosh(\pi\tilde{\omega})}{\sinh^2[\pi(\tilde{\omega}+i\epsilon)]}\,. \tag{3.36}$$

This solution can be simplified by subtracting a suitable amount of the previously found solution $\delta(\nu)$ so that the spectral function $d_0(\tilde{\omega})$ is replaced by

$$d_0'(\tilde{\omega}) \equiv \tilde{\omega}(\tilde{\omega}-i)\pi\frac{\nu^{-i\tilde{\omega}}\cosh(\pi\tilde{\omega})}{\sinh^2[\pi(\tilde{\omega}+i\epsilon)]}\,. \tag{3.37}$$

We denote the corresponding solution by $\Phi_0'(\nu)$.

4. The "in" and "out" states

4.1 Conformal primaries at the infrared fixed point

The $t \to -\infty$ limit of the time-dependent BCFT (2.12) is the unperturbed FZZT theory characterized by δ. The incoming scattering states are the states corresponding to the boundary operators $e^{iPX_0}\Phi^{\delta}_{|P|}$. As in the previous section we consider momenta $P = \mathcal{O}(\delta)$.

It was shown in [2] that the $t \to \infty$ limit can be described by the $\delta_* = -\delta$ FZZT boundary condition perturbed by a marginal oscillating term (2.25) interpreted as radiation. It is natural then to define the out-states as operators of the form $e^{iPX_0}\Phi^{-\delta}_{|P|}$. Since the time-dependent theory (2.12) is constructed as a perturbation in the Fock space (2.7) of the theory labeled by δ one needs to construct in that Fock space the operators $\Phi^{-\delta}_{|P|}$ out of the operators $\Phi^{\delta}_{|P|}$, $\Phi^{\delta}_{\vartheta}$. This problem is most elegantly solved by using the boundary RG flow generated by the relevant operator $\Phi^{\delta}_{\vartheta}$. This RG flow was studied in [1, 2]. The renormalized boundary action containing all near-marginal couplings generated by the flow has the form

$$S^{\rm ren}_{\rm Bd} = \int_{\mathbb{R}} dx \left(u\epsilon^{-\delta^2}\Phi^{\delta}_{\vartheta}(x) + \int_0^{\infty} dP\, \lambda(P)\epsilon^{P^2}\Phi^{\delta}_{P}(x)\right) \tag{4.1}$$

where ϵ is the regularization scale. The RG equations computed in the $\delta \ll 1$ conformal perturbation theory read [2]

$$\begin{aligned} \epsilon\frac{du}{d\epsilon} &= \delta^2 u - 2\delta w^2\,, \\ \epsilon\frac{d\lambda(P)}{d\epsilon} &= -P^2\lambda(P) - \frac{2P^2}{\pi(\delta^2+P^2)}w^2 \end{aligned} \tag{4.2}$$

where

$$w = u + \int_0^{\infty} dP\, \lambda(P)\,.$$

It was shown in [2] that the infrared fixed point of the above equation

$$u_* = 2\delta\,, \qquad \lambda_*(P) = -\frac{2\delta^2}{\pi(\delta^2+P^2)}$$

corresponds to the $\delta_* = -\delta$ FZZT conformal boundary condition.

In general if beta functions have the form:

$$\beta^j = \delta^j\lambda^j - \sum_{kl} C^j_{kl}\lambda^k\lambda^l \tag{4.3}$$

and $\lambda^j = \lambda^j_*$ is an IR fixed point, then

$$D^j_i \equiv (\partial_i\beta^j)(\lambda_*) \tag{4.4}$$

is the matrix of anomalous dimensions at the IR fixed point. Its spectrum of eigenvalues gives the IR fixed point anomalous dimensions and its eigenvectors the expressions for

IR fixed point conformal primaries via the UV ones[4] (which is sensible in the small δ expansion).

For the model at hand we find using (4.2) that the operator playing the role of the matrix D_i^j above acts as

$$\begin{aligned} \hat{D}\Phi_\vartheta^\delta &= \delta^2\Phi_\vartheta^\delta - 2\delta\Big(\int_0^\infty dQ\, f(Q)\Phi_Q^\delta + 2\delta\Phi_\vartheta^\delta\Big) \\ \hat{D}\Phi_P^\delta &= -P^2\Phi_P^\delta - 2\delta\Big(\int_0^\infty dQ\, f(Q)\Phi_Q^\delta + 2\delta\Phi_\vartheta^\delta\Big) \end{aligned} \tag{4.5}$$

where $f(Q)$ is given in (3.9).

To find the eigenvectors of operator $\hat{D}$ we write an ansatz

$$\Phi_P^* = \Phi_\vartheta^\delta + \int_0^\infty dQ\, K_P(Q)\Phi_Q^\delta\,. \tag{4.6}$$

The eigenvector equation

$$\hat{D}\,\Phi_P^* = -P^2\,\Phi_P^* \tag{4.7}$$

implies

$$\begin{aligned} K_P(Q)(Q^2-P^2) &= -\frac{\delta}{2}(p^2+1)f(Q)\,, \\ 1+\int_0^\infty dQ\, K_P(Q) &= \frac{1}{4}(p^2+1)\,. \end{aligned} \tag{4.8}$$

We find a distributional solution

$$K_P(Q) = g_p\delta(P-Q) - \frac{\delta(p^2+1)f(P)}{2(Q^2-P^2+i\epsilon P)} \tag{4.9}$$

The coefficient g_p is determined by substituting (4.9) into the second equation in (4.8). We obtain

$$g_p = \frac{1}{4}(p-i)^2\,. \tag{4.10}$$

We also checked that there are no solutions to the eigenvalue equation (4.7) for imaginary P which is consistent with the fact that in the IR fixed point there are no normalizable relevant operators.

We can check that the wavefunctions of operators (4.9), (4.10) have the correct asymptotics at $\phi_0 \to -\infty$. Assuming the $\phi_0 \to -\infty$ limit can be taken inside the integral over P in (4.6) using (2.11) and taking the appropriate residues we obtain the following asymptotic

$$\begin{aligned} \Psi_\vartheta(\phi_0) + \int_0^\infty dQ\, K_p(Q)\Psi_Q(\phi_0) &\sim \frac{1}{4}[(p+i)^2C_\delta(P)e^{iP\phi_0} + (p-i)^2C_\delta(-P)e^{-iP\phi_0}] \\ &= (p^2+1)\frac{1}{4}[C_{-\delta}(P)e^{iP\phi_0} + C_{-\delta}(-P)e^{-iP\phi_0}] \end{aligned} \tag{4.11}$$

[4]Of course this can be used to obtain only a subset of the primaries which are tangential to the RG flow in the vicinity of the IR fixed point

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This means that to the leading order in δ we can identify

$$\Phi_P^{-\delta} = \frac{4}{p^2+1}\Phi_P^* = \frac{4}{p^2+1}\Phi_\vartheta^\delta + \left(\frac{p-i}{p+i}\right)\Phi_{|P|}^\delta - 2\delta\int_0^\infty dQ \frac{f(Q)}{(Q^2-P^2+i\epsilon P)}\Phi_Q^\delta\,. \qquad (4.12)$$

Note the following simple properties

$$\Phi_P^{\pm\delta} = \Phi_{-P}^{\pm\delta} = (\Phi_P^{\pm\delta})^\dagger\,.$$

4.2 Conserved inner product and normalizations

An important property of the time evolution equations (3.11) is the existence of a conserved Klein-Gordon type inner product. For any two solutions $(\mu_i, \eta_i), i = 1, 2$ we define the inner product to be

$$\begin{aligned}\langle(\mu_1,\eta_1),(\mu_2,\eta_2)\rangle_{\rm KG} &= \frac{i}{2}\int_0^\infty dQ\, C_\delta(Q)C_\delta(-Q)(\mu_2^*\partial_t\mu_1 - \mu_1\partial_t\mu_2^*) + \frac{i}{2}d_\delta(\eta_2^*\partial_t\eta_1 - \eta_1\partial_t\eta_2^*)\\ &= \frac{i\pi\mu_r}{2\delta}\left[\int_0^\infty dQ\frac{\pi\delta(\delta^2+Q^2)}{Q^2}(\mu_2^*\partial_t\mu_1 - \mu_1\partial_t\mu_2^*) + (\eta_2^*\partial_t\eta_1 - \eta_1\partial_t\eta_2^*)\right].\end{aligned} \qquad (4.13)$$

It is straightforward to check using (3.11) that this inner product is t-independent. The weight function under the integral and the coefficient at the term with $\eta_{1,2}$ can be understood as a consequence of the two-point function normalizations (2.13), (2.14). We will use the same notation $\langle\Phi_{P_1}(\nu), \Phi_{P_2}(\nu)\rangle_{\rm KG}$ to define the inner product for operators (3.6) corresponding to solutions of (3.11), (3.12).

The operators $e^{iPX_0}\Phi_{|P|}^\delta$ are asymptotic solutions to (3.11) in the far past. Since the inner product (4.13) is conserved it can be evaluated on such operators. We choose a basis of normalized positive frequency "in" states to be given by operators

$$\mathcal{O}_P^{\rm in} = \frac{1}{|P|^{1/2}C_\delta(P)}\Phi_P^\delta e^{-iPX_0}\,,\ \ P>0\,. \qquad (4.14)$$

The hermitean conjugated operators form the basis of negative frequency "in" states:

$$\mathcal{O}_P^{\rm in*} = \frac{1}{|P|^{1/2}C_\delta(-P)}\Phi_P^\delta e^{iPX_0}\,,\ \ P>0\,. \qquad (4.15)$$

These operators satisfy the orthogonality conditions

$$\langle\mathcal{O}_{P_1}^{\rm in}, \mathcal{O}_{P_2}^{\rm in}\rangle_{\rm KG} = \delta(P_1-P_2)\,,\quad \langle\mathcal{O}_{P_1}^{\rm in*}, \mathcal{O}_{P_2}^{\rm in*}\rangle_{\rm KG} = -\delta(P_1-P_2)\,,\quad \langle\mathcal{O}_{P_1}^{\rm in*}, \mathcal{O}_{P_2}^{\rm in}\rangle_{\rm KG} = 0\,. \quad (4.16)$$

We are next going to show that operators $\Phi_{|P|}^{-\delta}e^{iPX_0}$ are asymptotic solutions to (3.11) at $t \to +\infty$. Substituting the asymptotic value $\delta = \lim_{t\to+\infty} W(t)$ for $W(t)$ we rewrite (3.11) in the following form

$$\partial_t^2\mu(t,P,Q) = \hat{D}\mu(t,P,Q) \equiv -Q^2\mu(t,P,Q) - 2\delta f(Q)(\eta(t,P) + \int_0^\infty dQ'\mu(t,P,Q'))\,,$$
$$\partial_t^2\eta(t,P) = \hat{D}\eta(t,P) \equiv \delta^2\eta(t,P) - 4\delta f(Q)(\eta(t,P) + \int_0^\infty dQ'\mu(t,P,Q'))\,. \tag{4.17}$$

The operator $\hat{D}$ defined on the right hand sides can be recognized as the dual action on the coupling constants of the dilatation operator (4.5). Thus by (4.7), (4.12) the operators $\Phi_{|P|}^{-\delta}e^{iPX_0}$ are indeed solutions of the wave equation (3.11) at $t\to+\infty$.

We define a basis of positive frequency "out" states

$$\mathcal{O}_P^{\rm out} = \left(\frac{p-i}{p+i}\right)\frac{1}{|P|^{1/2}C_\delta(P)}\Phi_P^{-\delta}e^{-iPX_0}\,,\quad P>0 \tag{4.18}$$

and a hermitean conjugated basis $\mathcal{O}_P^{\rm out*}$ of negative frequency states. The normalization factor is chosen so that

$$\langle\mathcal{O}_{P_1}^{\rm out},\mathcal{O}_{P_2}^{\rm out}\rangle_{\rm KG} = \delta(P_1-P_2)\,,\quad \langle\mathcal{O}_{P_1}^{\rm out*},\mathcal{O}_{P_2}^{\rm out*}\rangle_{\rm KG} = -\delta(P_1-P_2)\,,$$
$$\langle\mathcal{O}_{P_1}^{\rm out*},\mathcal{O}_{P_2}^{\rm out}\rangle_{\rm KG} = 0\,. \tag{4.19}$$

This is checked by direct computation substituting the coefficients from (4.12) into (4.13). The particular choice of phase in (4.18) is done for convenience.

4.3 The $t\to+\infty$ asymptotics

One can compute the $t\to+\infty$ asymptotics of the time-dependent vertex operators (3.6). What we formally mean by taking such a limit is taking first the limit of the analytically continued solutions to the wave equations (3.20). The asymptotic coupling constants proportional to e^{iQt} can then be coupled to operators $e^{iQX_0}\Phi_{Q'}^{\delta}$ giving thus the operator-valued limit[5]. The desired asymptotic can be obtained either by using the appropriate asymptotic expansion for the Lerch phi-function (hypergeometric function) or by taking residues in the complex ω-plane in the spectral formulas (3.24). One obtains

$$\mu(t,P,Q) \underset{t\to\infty}{\sim} e^{iPt}\xi_0\left[\delta(P-Q) - \frac{2f(Q)\delta}{(Q^2-P^2+i\epsilon P)}\left(\frac{p+i}{p-i}\right)\right] + \tag{4.20}$$
$$+\frac{\xi_0\pi f(Q)}{\delta p(1+ip)}\left[e^{-iQt}\frac{(1+iq)\nu^{-i(q+p)}}{\sinh[\pi(q+p-i\epsilon)]} + e^{iQt}\frac{(1-iq)\nu^{-i(p-q)}}{\sinh[\pi(-q+p-i\epsilon)]}\right].$$

Note that when working with the first spectral expression (3.24) we have to take into account two $i\epsilon$ contour prescriptions: one explicitly present in (3.24) and another one in the Fourier transform of W (3.26). The $t\to\infty$ limit depends on the difference of two

[5]Putting it a bit differently (c.f. the discussion in [2]) one can speak of a limit for the wave functions obtained as an overlap of the states corresponding to (3.6) with the states $\langle\phi_0^{\rm op}|\otimes\langle t|$ as in (2.10) and then invoke the state-operator correspondence.

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epsilons. The choice of the sign is however insignificant as the complete expression is non-singular at $p = q$. The particular choice of sign was made for an easy comparison with (4.12). Similarly we obtain

$$\eta(t,P) \underset{t\to\infty}{\sim} \frac{4}{(p-i)^2} e^{iPt} \,. \tag{4.21}$$

Comparing (3.12) with (4.14) we can fix the normalization constant

$$\xi_0 = \xi_0(P) = \frac{1}{|P|^{1/2} C_\delta(-P)} \tag{4.22}$$

so that in the far past the operator $\Phi_{-P}(\nu)$, $P > 0$ approaches the normalized positive frequency in operator $\mathcal{O}_P^{\rm in}$:

$$\Phi_{-P}(\nu) \underset{t\to-\infty}{\sim} \mathcal{O}_P^{\rm in} \,, \;\; P > 0 \,. \tag{4.23}$$

From now on we will assume that $P > 0$ and will work with the vertex operator $\Phi_{-P}(\nu)$. Combining (4.20) and (4.21) together and taking into account (4.12), (4.18) we obtain

$$\begin{aligned} \Phi_{-P}(\nu) \underset{t\to\infty}{\sim} \; & \mathcal{O}_P^{\rm out} + \frac{\xi_0(-P)\pi}{\delta p(1-ip)} \times \\ & \times \int_0^\infty dQ f(Q) \Phi_Q^\delta \left[e^{-iQX_0} \frac{(1+iq)\nu^{i(p-q)}}{\sinh[\pi(p-q+i\epsilon)]} + e^{iQX_0} \frac{(1-iq)\nu^{i(p+q)}}{\sinh[\pi(p+q+i\epsilon)]} \right] . \end{aligned} \tag{4.24}$$

For the zero momentum solutions $\delta(\nu)$ and $\Phi_0(\nu)$ one finds the following asymptotics

$$\delta(\nu) \underset{t\to\infty}{\sim} \int_0^\infty dQ \frac{2q^2 \Phi_Q^\delta}{\sinh(\pi q)} \left[\frac{e^{-iQX_0}\nu^{-iq}}{1-iq} - \frac{e^{iQX_0}\nu^{iq}}{1+iq} \right] , \tag{4.25}$$

$$\begin{aligned} \Phi_0(\nu) \underset{t\to\infty}{\sim} \; & -\Phi_0^{-\delta} \\ & -\frac{2\pi}{\delta} \int_0^\infty dQ q^2 \cosh(\pi q) \Phi_Q^\delta \left[\frac{e^{-iQX_0}\nu^{-iq}}{(1-iq)\sinh^2[\pi(q+i\epsilon)]} + \frac{e^{iQX_0}\nu^{iq}}{(1+iq)\sinh^2[\pi(q-i\epsilon)]} \right] . \end{aligned} \tag{4.26}$$

5. Solutions asymptoting to positive frequency out states

The $t \to \pm\infty$ asymptotics of the zero mode $\delta(\nu)$ present us the following apparent problem. On the one hand the $t \to +\infty$ asymptotic given in (4.25) means that the asymptotic overlaps

$$\langle \mathcal{O}_Q^{\rm out}, \delta(\nu) \rangle_{\rm KG}$$

are nonvanishing. On the other hand given a solution $\Psi_{-P}(\nu)$ that at $t \to +\infty$ approaches $\mathcal{O}_P^{\rm out}$, its overlap with $\delta(\nu)$ would be zero provided the $t \to -\infty$ asymptotic of $\Psi_{-P}(\nu)$ is bounded. This is because the $\delta(\nu)$ solution asymptotes to zero at $t \to -\infty$. We are forced to conclude that either the solutions approaching $\mathcal{O}_P^{\rm out}$ in the infinite future do not exist at all or, if they exist, they blow up at $t \to -\infty$. We will show in this section that it is the second option that is realized.

A solution $\Psi_P(\nu)$ that asymptotically approaches $\mathcal{O}_P^{\mathrm{out}*}$ can be constructed as follows. We start by looking at a solution at large positive values of t in the form of series expansions

$$\tilde{\mu}(t,Q,P) = \sum_{n=0}^{\infty} \tilde{\mu}_n e^{(iP-n\delta)t}\,, \quad \tilde{\eta}(t,P) = \sum_{n=0}^{\infty} \tilde{\eta}_n e^{(iP-n\delta)t} \tag{5.1}$$

with

$$\tilde{\mu}_0 = \frac{4\tilde{h}_0}{p^2+1} K_P(Q) = \tilde{h}_0 \left[\left(\frac{p-i}{p+i} \right) \delta(Q-|P|) - \frac{2\delta f(Q)}{Q^2-P^2+i\epsilon P} \right], \tag{5.2}$$

$$\tilde{\eta}_0 = \frac{4\tilde{h}_0}{p^2+1} \tag{5.3}$$

where $\tilde{h}_0$ is a normalization factor and $P>0$. The coefficients $\tilde{\mu}_0$, $\tilde{\eta}_0$ above are chosen so that

$$\Psi_P(\nu) \equiv \tilde{\eta}(t,P)\Phi_\vartheta^\delta + \int_0^{+\infty} dQ\,\tilde{\mu}(t,Q,P)\Phi_Q^\delta \underset{t\to\infty}{\sim} \tilde{h}_0 \Phi_{|P|}^{-\delta} e^{iPt}\,. \tag{5.4}$$

Choosing the normalization factor

$$\tilde{h}_0 = \tilde{h}_0(P) = \left(\frac{p+i}{p-i} \right) \frac{1}{|P|^{1/2} C_\delta(-P)} \tag{5.5}$$

we obtain

$$\Psi_P(\nu) \underset{t\to\infty}{\sim} \mathcal{O}_P^{\mathrm{out}*}\,, \quad \Psi_{-P}(\nu) \underset{t\to\infty}{\sim} \mathcal{O}_P^{\mathrm{out}}\,. \tag{5.6}$$

Substituting the expansions (5.1) into the wave equations (3.11) we obtain the following set of equations

$$\begin{aligned} [Q^2 + (iP-n\delta)^2]\tilde{\mu}_n &= -2f(Q) \sum_{l=0}^{n} \tilde{W}_l \tilde{h}_{n-l}\,, \\ [(iP-n\delta)^2 - \delta^2]\tilde{\eta}_n &= -4\delta \sum_{l=0}^{n} \tilde{W}_l \tilde{h}_{n-l} \end{aligned} \tag{5.7}$$

where

$$\tilde{h}_n = \tilde{\xi}_n + \tilde{\eta}_n \equiv \int_0^{+\infty} dQ\, \mu_n(Q) + \tilde{\eta}_n \tag{5.8}$$

and $\tilde{W}_l = \delta(-\nu)^{-l}$ are coefficients in the expansion

$$W(t) = \sum_{l=0}^{\infty} \tilde{W}_l e^{-n\delta t}\,.$$

We solve the system (5.7) by similar steps to those we did in section 3.2 solving (3.7). The first equation in (5.7) together with (5.2) imply

$$\tilde{\mu}_n = -\frac{2f(Q)(W\cdot h)_n}{Q^2 + (i(P-i\epsilon)-n\delta)^2} + \delta_{n,0}\tilde{h}_0 \left(\frac{p-i}{p+i} \right) \delta(Q-|P|) \tag{5.9}$$

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where for brevity we used the notation

$$(W\cdot h)_n \equiv \sum_{l=0}^{n}\tilde W_l \tilde h_{n-l}\,.$$

Integrating the last equation over Q we obtain

$$(\delta(n+1)-iP)\tilde\xi_n = -2(W\cdot h)_n - i\delta_{n,0}\tilde h_0\frac{(p+i)^2}{p-i}\delta\,. \tag{5.10}$$

Taking the appropriate linear combinations of equations (5.7), (5.10) we obtain

$$\begin{aligned}
\tilde\mu_n &= \frac{\delta f(Q)\tilde h_n(n-1+ip)}{Q^2+(iP+\epsilon-n\delta)^2}+\delta_{n,0}\left[\tilde\mu_0+\frac{\delta f(Q)\tilde h_0(1-ip)}{Q^2-(iP+\epsilon)^2}\right],\\
\tilde\eta_n &= \frac{2\tilde h_n}{n+1-ip}-\frac{2i}{p-i}\tilde h_0\delta_{n,0}\,, \qquad (5.11)\\
(iP+\delta(1-n))\tilde h_n &= 2(W\cdot h)_n + i\delta_{n,0}\tilde h_0(p+i)\delta\,. \qquad (5.12)
\end{aligned}$$

The last equation is solved in terms of hypergeometric series

$$\tilde h(t)=\tilde h_0\left[\left(\frac{p+i}{p-i}\right)-\frac{2i}{p-i}{}_2F_1(1,-1-ip,2-ip;-\tilde x)\right]e^{iPt}\,. \tag{5.13}$$

where $\tilde x = \nu^{-1}e^{-\delta t}$. Substituting the coefficients $\tilde h_n$ into (5.11) we obtain the following expressions for $\tilde\mu(t,Q,P)$, $\tilde\eta(t,P)$ in terms of Lerch phi-functions

$$\begin{aligned}
\tilde\mu(t,Q,P) &= \delta(Q-|P|)\tilde h_0\left(\frac{p-i}{p+i}\right)e^{iPt}+\tilde h_0\frac{f(Q)}{Q}ip(p+i)e^{iPt}\left[-\frac{\Phi(-\tilde x,1,i(q-p))}{q(q+i)}\right.\\
&\left.-\frac{2i}{q}\Phi(-\tilde x,1,-ip)+\frac{\Phi(-\tilde x,1,-i(q+p))}{q(q-i)}+\frac{2iq}{1+q^2}\Phi(-\tilde x,1,1-ip)\right], \qquad (5.14)\\
\tilde\eta(t,P) &= 4\tilde h_0 p(p+i)e^{iPt}\left[\frac{1}{2}\Phi(-\tilde x,2,1-ip)-\Phi(-\tilde x,1,-ip)+\right.\\
&\left.+\frac{3}{4}\Phi(-\tilde x,1,1-ip)+\frac{1}{4}\Phi(-\tilde x,1,-1-ip)\right]. \qquad (5.15)
\end{aligned}$$

An asymptotic expansion of $\Psi_P(\nu)$ in the $t\to-\infty$ region can be obtained from expressions (5.14), (5.15) using relation (3.23) for the functions $\Phi(-\tilde x,1,v)$ and the following integral expression

$$\Phi(-\tilde x,2,1-ip)=\frac{1}{2i}\int_{-\infty}^{+\infty}d\omega\frac{\nu^{-i\omega}e^{-i\omega\delta t}}{(\omega-p-i)^2\sinh[\pi(\omega+i\epsilon)]}\,. \tag{5.16}$$

The leading terms in the asymptotic read

$$\begin{aligned}
\tilde\mu(t,Q,P) &\underset{t\to-\infty}{\sim} \tilde h_0\left(\frac{p-i}{p+i}\right)\delta(Q-|P|)e^{iPt}-\\
&-\tilde h_0\pi p(p+i)\frac{f(Q)}{\delta q^2}\left[\frac{\nu^{i(q-p)}e^{iQt}}{(q+i)\sinh[\pi(q-p)]}+\frac{\nu^{-i(q+p)}e^{-iQt}}{(q-i)\sinh[\pi(q+p)]}\right]+\\
&-2\frac{f(Q)}{\delta q^2}\alpha(p)\,. \qquad (5.17)\\
\tilde\eta(t,P) &\underset{t\to-\infty}{\sim} \alpha(p)(\nu^{-1}e^{-\delta t}+4+\mathcal{O}(e^{\delta t})) \qquad (5.18)
\end{aligned}$$

where

$$\alpha(p) = \tilde{h}_0(P)\frac{\pi p(1-ip)\nu^{-ip}}{\sinh(\pi p)} = -i\left(\frac{p+i}{p-i}\right)\frac{|p|^{3/2}\nu^{-ip}}{(\mu_r\delta)^{1/2}\sinh(\pi p)}\,. \tag{5.19}$$

The first term in (5.18) is exponentially divergent in the $t\to-\infty$ limit and therefore has a nonzero overlap with the $\delta(\nu)$ solution. This overlap matches with the one that can be computed from the expansions in the $t\to+\infty$ region using (5.18).

6. Bogolyubov transformations

To obtain the Bogolyubov coefficients we start by recasting asymptotics 4.24 into a form

$$\Phi_{-P}(\nu) \underset{t\to\infty}{\sim} \int_0^\infty dQ(\beta_P(Q)\mathcal{O}_Q^{\text{out}*} + \alpha_P(Q)\mathcal{O}_Q^{\text{out}}) \tag{6.1}$$

where $\beta_P(Q)$ and $\alpha_P(Q)$ are Bogolyubov coefficients. Using the completeness of the out-basis (4.18) and the inner products (4.19) we can evaluate the Bogolyubov coefficients at hand via the asymptotics of the inner-products

$$\alpha_P(Q) \underset{t\to\infty}{\sim} \langle\Phi_{-P}(\nu),\mathcal{O}_Q^{\text{out}}\rangle_{\text{KG}}\,,\quad \beta_P(Q) \underset{t\to\infty}{\sim} -\langle\Phi_{-P}(\nu),\mathcal{O}_Q^{\text{out}*}\rangle_{\text{KG}}\,. \tag{6.2}$$

It is technically simpler to compute overlaps of $\mathcal{O}_Q^{\text{out}}$ with the asymptotic (4.24) dropping any additional terms exponentially suppressed as $t\to\infty$. After the substitution of (4.24), (4.18) into (4.13) the computation reduces to finding the asymptotic value of the following integral

$$I \equiv \int_{-\infty}^{+\infty} ds\frac{s^2(q-s)}{(1-is)(s^2-q^2\pm i\epsilon)}\frac{e^{-i(s+q)\delta t}\nu^{i(p-s)}}{\sinh[\pi(s-p-i\epsilon')]} \tag{6.3}$$

where $+i\epsilon$ occurs in the computation of $\alpha_P(Q)$ and $-i\epsilon$ occurs for $\beta_P(Q)$. For $t\to+\infty$ this integral can be evaluated by taking residues in the region $\mathrm{Im}s\le 0$. Contributions from the residues located away from the real line are suppressed as $\exp(-n\delta t)$ and should be dropped. Evaluating the residues on the real line eventually yields the following expressions for the Bogolyubov coefficients

$$\begin{aligned}\alpha_P(Q) &= \delta(P-Q) - \frac{2|q|^{3/2}}{\delta|p|^{1/2}(p^2+1)}\left(\frac{q+i}{q-i}\right)\frac{\nu^{i(p-q)}}{\sinh[\pi(p-q+i\epsilon)]}\,,\\ \beta_P(Q) &= \frac{2|q|^{3/2}}{\delta|p|^{1/2}(p^2+1)}\left(\frac{q-i}{q+i}\right)\frac{\nu^{i(p+q)}}{\sinh[\pi(p+q+i\epsilon)]}\,.\end{aligned} \tag{6.4}$$

Conservation of the inner product (4.13) together with asymptotics (4.23), (6.1) and formula (4.16) imply the following pair of relations for Bogolyubov coefficients

$$\begin{aligned}&\int_0^\infty dQ[\alpha_{P_1}(Q)\alpha^*_{P_2}(Q) - \beta_{P_1}(Q)\beta^*_{P_2}(Q)] = \delta(P_1-P_2)\,,\\ &\int_0^\infty dQ[\alpha_{P_1}(Q)\beta_{P_2}(Q) - \alpha_{P_2}(Q)\beta_{P_1}(Q)] = 0\,.\end{aligned} \tag{6.5}$$

These relations can be checked for the coefficients (6.4) by direct computation. The details of this computation are relegated to appendix A.

A direct computation leads to another pair of relations

$$\begin{aligned}\int_0^\infty dP[\alpha^*_P(Q_1)\alpha_P(Q_2) - \beta_P(Q_1)\beta^*_P(Q_2)] &= \delta(Q_1 - Q_2) + d(Q_1, Q_2)\,,\\ \int_0^\infty dP[\alpha^*_P(Q_1)\beta_P(Q_2) - \alpha^*_P(Q_2)\beta_P(Q_1)] &= -d(Q_1, -Q_2)\end{aligned} \tag{6.6}$$

where an explicit expression for $d(Q_1, Q_2)$ is given in (A.6). This pair of relations deviates from the ones one could have expected naively - the ones in which the function $d(Q_1, Q_2)$ is identically zero. This signals incompleteness of the set of incoming solutions $\Phi_{-P}(\nu)$. Let us recall the logic of derivation of the standard relations with vanishing $d(Q_1, Q_2)$. The inner products (6.2) and the completeness of the asymptotic solutions $\mathcal{O}^{\rm out}_P$ imply that the solution

$$\chi_Q(\nu) \equiv \int_0^\infty dP\,[\alpha^*_P(Q)\Phi_{-P}(\nu) - \beta_P(Q)\Phi_P(\nu)] \tag{6.7}$$

asymptotically at $t \to +\infty$ has the same overlaps with all $\Phi_{-P}(\nu)$ as $\mathcal{O}^{\rm out}_Q$. If the set of solutions $\Phi_{-P}(Q)$ was complete we would conclude that the solution $\chi_Q(\nu)$ asymptotically approaches $\mathcal{O}^{\rm out}_Q$ at $t \to +\infty$ (and thus should be identified with the solution $\Psi_{-P}(\nu)$ considered before). The inner product conservation would then imply $\langle\chi_{Q_1}(\nu), \chi_{Q_2}(\nu)\rangle_{\rm KG} = \delta(Q_1 - Q_2)$, $\langle\chi^*_{Q_1}(\nu), \chi_{Q_2}(\nu)\rangle_{\rm KG} = 0$ which in its turn would imply the standard relations (6.6) with vanishing $d(Q_1, Q_2)$. We do know however from the considerations in section 5 that the set of solutions $\Phi_{-P}(Q)$ is incomplete due to the existence of solutions blowing up in the infinite past. In that section we constructed the solution $\Psi_{-P}(\nu)$ that approaches $\mathcal{O}^{\rm out}_P$ explicitly and saw that its asymptotic in the infinite past blows up.

The complete asymptotic expansion in the $t \to -\infty$ region obtained using (3.23), (5.16) together with the asymptotic expansion for $\chi_P(\nu)$ in the same region can be used to identify the $\Psi_{-P}(\nu)$ solution with the following linear combination of solutions

$$\Psi_{-P}(\nu) = \chi_P(\nu) + \alpha(-p)\tilde{\delta}(\nu) + \beta(-p)\delta(\nu) \tag{6.8}$$

where

$$\tilde{\delta}(\nu) = \Phi'_0(\nu) + \Phi^{\delta}_{\vartheta}\nu^{-1}e^{-\delta X_0} + \Phi_0^{-\delta}\,, \tag{6.9}$$

$$\beta(p) = -\frac{\pi}{\delta}\alpha(p)\coth(\pi p)\,. \tag{6.10}$$

The above expansion means that in order to obtain a complete set of scattering "out" states one needs to include, in addition to solutions constructed in section 3, a single solution $\tilde{\delta}(\nu)$ blowing up in the $t \to -\infty$ limit.

We define two sets of solutions

$$\mathcal{S}^{\rm in} = \{\Phi_P(\nu)\,,\;\; P \neq 0; \delta(\nu); \tilde{\delta}(\nu)\}\,,\quad \mathcal{S}^{\rm out} = \{\Psi_P(\nu)\,,\;\; P \in \mathbb{R}\}\,.$$

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The inner products between the bases of solutions asymptoting to the "in" and "out" states are

$$\begin{aligned}
\langle \Phi_P(\nu), \Phi_Q(\nu)\rangle_{\rm KG} &= -{\rm sgn}(P)\delta(P-Q)\,, \\
\langle \Psi_P(\nu), \Psi_Q(\nu)\rangle_{\rm KG} &= -{\rm sgn}(P)\delta(P-Q)\,, \\
\langle \Phi_P(\nu), \delta(\nu)\rangle_{\rm KG} &= \langle \Phi_P(\nu), \tilde{\delta}(\nu)\rangle_{\rm KG} = 0 \quad \text{for } P\neq 0 \\
\langle \delta(\nu), \tilde{\delta}(\nu)\rangle_{\rm KG} &= -2\pi\mu_r\delta\,.
\end{aligned} \tag{6.11}$$

The functions $\alpha(-p)$, $\beta(-p)$ given in (5.19), (6.10) are additional Bogolyubov coefficients. Substituting the expansion (6.8) into the second identity in (6.11) gives the second pair of relations for Bogolyubov coefficients (6.6) where the function $d(q_1,q_2)$ is

$$d(Q_1,Q_2) = \langle \delta(\nu), \tilde{\delta}(\nu)\rangle_{\rm KG}[\alpha(-q_1)\beta(q_2) - \alpha(q_2)\beta(-q_1)]$$

in accordance with (A.6). Thus the extra term on the right hand side of (6.6) indeed accounts for the blowing up solution $\tilde{\delta}(\nu)$.

The following additional relations can be shown to be true

$$\begin{aligned}
\int_0^\infty dQ\,[\alpha^*_P(Q)\beta(q) + \beta^*_P(Q)\beta(-q)] &= 0\,, \\
\int_0^\infty dQ\,[\alpha^*_P(Q)\alpha(q) + \beta^*_P(Q)\alpha(-q)] &= 0\,, \\
\int_0^\infty dQ\,[\alpha(-q)\beta(q) - \alpha(q)\beta(-q)] &= (\tilde{\mu})^{-1}\,.
\end{aligned} \tag{6.12}$$

Here and elsewhere $\tilde{\mu}$ stands for the combination

$$\tilde{\mu} = 2\pi\mu_r\delta\,. \tag{6.13}$$

The expressions for the modes $\mathcal{S}^{\rm in}$ in terms of the modes $\mathcal{S}^{\rm out}$ can be found using the inner products (6.2), (6.11) and formula (6.8). We derive

$$\begin{aligned}
\Phi_{-P}(\nu) &= \int_0^\infty dQ(\beta_P(Q)\Psi_Q(\nu) + \alpha_P(Q)\Psi_{-Q}(\nu))\,, \\
\delta(\nu) &= \tilde{\mu}\int_0^\infty dP[\alpha(-p)\Psi_P(\nu) - \alpha(p)\Psi_{-P}(\nu)]\,, \\
\tilde{\delta}(\nu) &= \tilde{\mu}\int_0^\infty dP[\beta(p)\Psi_{-P}(\nu) - \beta(-p)\Psi_P(\nu)]\,.
\end{aligned} \tag{6.14}$$

The expressions (6.7), (6.8), (6.14) define (inverse to each other) Bogolyubov transformations between the two sets of modes: $\mathcal{S}^{\rm in}$ and $\mathcal{S}^{\rm out}$. The corresponding Bogolyubov coefficients are all expressed via the functions $\alpha_P(Q), \beta_P(Q), \alpha(p), \beta(q)$, and satisfy the standard unitarity relations.

The inner product matrix for the basis $\mathcal{S}^{\rm in}$ can be completely diagonalized by introducing

$$\delta_+ = (\tilde{\delta}(\nu) + \delta(\nu))(2\tilde{\mu})^{-1/2}\,, \quad \delta_- = (\tilde{\delta}(\nu) - \delta(\nu))(2\tilde{\mu})^{-1/2}\,. \tag{6.15}$$

The modes δ_+, δ_- are complex conjugates of each other. One has

$$\langle \delta_+, \delta_+ \rangle_{\rm KG} = -1\,, \quad \langle \delta_-, \delta_- \rangle_{\rm KG} = 1\,, \quad \langle \delta_+, \delta_- \rangle_{\rm KG} = 0\,. \tag{6.16}$$

Let us introduce a linear operator A that corresponds to a block of the Bogolyubov transformation (6.8), (6.14) that maps the positive frequency modes $\Psi_{-P}(\nu)\,,\ P \geq 0$ to the modes $\Phi_{-P}(\nu)\,, P > 0\,,\ \delta_-$. It can be written as a block matrix

$$A = \begin{bmatrix} \alpha_P(Q) \\ \alpha(q) + \beta(q) \end{bmatrix}. \tag{6.17}$$

Here Q is a column's label. Similarly the block mapping the negative frequency out states $\Psi_{-P}(\nu)$, $P \leq 0$ to the modes $\Phi_{-P}(\nu)\,, P > 0\,,\ \delta_-$ is given by a block matrix

$$B = \begin{bmatrix} \beta_P(Q) \\ \alpha(-q) + \beta(-q) \end{bmatrix}. \tag{6.18}$$

In terms of the operators A, B the first relation in (6.6) can be rewritten as

$$A^\dagger A = I + BB^\dagger\,. \tag{6.19}$$

This implies that the operator A has the bounded inverse (e.g. see [19] chapter 2, section 4.2). The operator A^{-1} has a block structure

$$A^{-1} = \begin{bmatrix} \gamma_Q(P) & \gamma(q) \end{bmatrix} \tag{6.20}$$

where rows are labeled by Q . In terms of the block entries we obtain the following set of relations which will be used later

$$\begin{aligned}
\int_0^\infty dP\, \gamma_{Q_1}(P)\alpha_P(Q_2) &= \delta(Q_1 - Q_2) - \gamma(q_1)(\alpha(q_2) + \beta(q_2))\,, \\
\int_0^\infty dQ\, \alpha_{P_1}(Q)\gamma_Q(P_2) &= \delta(P_1 - P_2)\,, \\
\int_0^\infty dQ\, \alpha_P(Q)\gamma(q) &= 0\,, \\
\int_0^\infty dQ\, (\alpha(q) + \beta(q))\gamma_Q(P) &= 0\,, \\
\int_0^\infty dQ\, (\alpha(q) + \beta(q))\gamma(q) &= 1\,.
\end{aligned} \tag{6.21}$$

7. Secondary quantization

Let us discuss now what our findings imply for a secondary quantization of the wave equations (3.11). A quantum field $\hat{\Phi}$ can be decomposed in two ways as

$$
\begin{aligned}
\hat{\Phi} &= \int_0^\infty dP\, [\Phi_{-P}(\nu) a_P^{\text{in}} + \Phi_P(\nu) a_P^{\text{in}\dagger}] + \tilde{\mu}^{-1/2}(\hat{q}\tilde{\delta}(\nu) - i\hat{p}\delta(\nu)) \,, \\
\hat{\Phi} &= \int_0^\infty dP\, [\Psi_{-P}(\nu) a_P^{\text{out}} + \Psi_P(\nu) a_P^{\text{out}\dagger}]
\end{aligned}
\tag{7.1}
$$

where the creation and annihilation operators satisfy the canonical commutation relations

$$
\begin{aligned}
[a_{P_1}^{\text{in}}, a_{P_2}^{\text{in}\dagger}] &= \delta(P_1 - P_2)\,, & [a_{P_1}^{\text{in}}, a_{P_2}^{\text{in}}] &= 0 \\
[a_{P_1}^{\text{out}}, a_{P_2}^{\text{out}\dagger}] &= \delta(P_1 - P_2)\,, & [a_{P_1}^{\text{out}}, a_{P_2}^{\text{out}}] &= 0\,.
\end{aligned}
\tag{7.2}
$$

The operator modes $\hat{q}$ and $\hat{p}$ are hermitean and the commutation relations involving these operators can be easily determined. Recall that the symplectic form on the space of classical solutions can be defined as

$$
\Omega(\Phi_1, \Phi_2) = \langle \Phi_1, \Phi_2^* \rangle_{\text{KG}}\,. \tag{7.3}
$$

Following the rules of second quantization with this symplectic form we obtain from (6.11), (6.8)

$$
[\hat{q}, \hat{p}] = i\,, \quad [\hat{p}, a_P^{\text{in}}] = 0\,, \quad [\hat{q}, a_P^{\text{in}}] = 0\,. \tag{7.4}
$$

Substituting (6.8) into the second equation in (7.1) we obtain

$$
\begin{aligned}
a_P^{\text{in}} &= \int_\epsilon^\infty dQ\, [\alpha_P^*(Q) a_Q^{\text{out}} - \beta_P^*(Q) a_Q^{\text{out}\dagger}]\,, \\
\hat{p} &= i\tilde{\mu}^{1/2} \int_0^\infty dP\, [\beta(-p) a_P^{\text{out}} + \beta(p) a_P^{\text{out}\dagger}]\,, \\
\hat{q} &= \tilde{\mu}^{1/2} \int_0^\infty dP\, [\alpha(-p) a_P^{\text{out}} + \alpha(p) a_P^{\text{out}\dagger}]\,.
\end{aligned}
\tag{7.5}
$$

The inverse set of linear relations is found by substituting (6.14) into the first equation in (7.1). We have

$$
a_P^{\text{out}} = \int_0^\infty dQ\, [\alpha_Q(P) a_Q^{\text{in}} + \beta_Q^*(P) a_Q^{\text{in}\dagger}] + \tilde{\mu}^{1/2}(\beta(p)\hat{q} + i\alpha(p)\hat{p})\,. \tag{7.6}
$$

Note that the modes a_P^{in}, a_P^{out} each describes a harmonic oscillator with frequency P in the respective asymptotic regions. It is straightforward to define the "out" vacuum $|0\rangle_{\text{out}}$ as a state annihilated by all a_P^{out} operators. To define the "in" vacuum $|0\rangle_{\text{in}}$ it is

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natural to require $a_P^{\rm in}|0\rangle_{\rm in} = 0$ for all $P > 0$. In addition to this we need to specify how the operators $\hat{q}$, $\hat{p}$ act on $|0\rangle_{\rm in}$. The modes $\hat{q}$, $\hat{p}$ are canonical variables asymptotically describing a quantized upside-down harmonic oscillator. The last one is classically described by the equation

$$\partial_t^2 \eta = \delta^2 \eta \tag{7.7}$$

which is the asymptotic equation of motion for the mode η in the infinite past. The corresponding quantum Hamiltonian is unbounded and there is no natural choice of the incoming state for this system. We will discuss the physics of this mode further in section 10. For now we record that there is an essential ambiguity in defining $|0\rangle_{\rm in}$.

However it is not hard to see that there is a large class of pair creation amplitudes independent of the choice of initial wavefunction for modes $\hat{q}$, $\hat{p}$. Apply both sides of the first equation in (7.5) to the "in" vacuum. We obtain

$$\int_0^\infty dQ\, \alpha_P^*(Q) a_Q^{\rm out}|0\rangle_{\rm in} = \int_0^\infty dQ\, \beta_P^*(Q) a_Q^{{\rm out}\dagger}|0\rangle_{\rm in}\,.$$

Using this relation and the canonical commutation relations we obtain

$$\int_0^\infty dQ_1 \int_0^\infty dQ_2\, \alpha_{P_1}^*(Q_1)\alpha_{P_2}^*(Q_2)\,{}_{\rm out}\langle Q_1, Q_2|0\rangle_{\rm in} = \int_0^\infty dQ\, \alpha_{P_1}^*(Q)\beta_{P_2}^*(Q) \tag{7.8}$$

where ${}_{\rm out}\langle Q_1, Q_2|$ is the bra vector conjugated to[6] $|Q_1, Q_2\rangle_{\rm out} = a_{Q_1}^{{\rm out}\dagger} a_{Q_1}^{{\rm out}\dagger}|0\rangle_{\rm out}$. Formula (7.8) means that a creation amplitude for two "out" particles whose wave functions are of the form

$$\psi(Q) = \int_0^\infty dP\, \phi(P)\alpha_P(Q) \tag{7.9}$$

with any sensible weight function $\phi(P)$, are independent of how the action of the modes $\hat{q}$, $\hat{p}$ is defined on $|0\rangle_{\rm in}$. Such amplitudes are all expressible via the "out" wave functions and the Bogolyubov coefficients $\alpha_P(Q)$ and $\beta_P(Q)$. A natural question arises - how big is the space of all such functions? This is measured by the dimension of the cokernel of the operator defined in (7.9). The first relation in (6.21) implies that the cokernel of operator (7.9) has at most dimension one. Thus almost all pair creation amplitudes are independent of the details of the initial state of the tachyonic mode η. In the second quantized approach these amplitudes are expressed via Bogolyubov coefficients (7.8). In the next section we will show that pair creation amplitudes (7.8) can be obtained by computing an appropriate string theory two-point function.

[6] We chose to normalize our particle states so that their inner products involve only appropriate delta functions. Alternatively one may wish the normalizations to be invariant under the asymptotic Lorentz transformations. In that case one should include the factors $\sqrt{|P|}$ with every creation operator $a_P^{{\rm in}\dagger}$, $a_P^{{\rm out}\dagger}$. All formulas we obtain can be trivially generalized to include such factors.

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8. String theory two point function

As was said in the introduction it is natural to expect that string theory two point functions in time-dependent backgrounds are related to pair creation amplitudes which are expressible via Bogolyubov coefficients (see also [8]). We would like to check this relationship for the set of amplitudes described in the previous section. We will show that for a properly defined string two point function the following relation holds

$$
\begin{aligned}
\frac{1}{2}\langle\Phi_{-P_1}(\nu)\Phi_{-P_2}(\nu)\rangle_{\rm str} &= \int_0^\infty dQ_1\int_0^\infty dQ_2\,\alpha_{P_1}(Q_1)\alpha_{P_2}(Q_2)_{\rm in}\langle 0|Q_1,Q_2\rangle_{\rm out} \\
&= \int_0^\infty dQ\,\alpha_{P_1}(Q)\beta_{P_2}(Q)\,.
\end{aligned}
\tag{8.1}
$$

Defining a two point amplitude in string theory involves fixing the $SL(2,\mathbb{R})$ modular symmetry. Fixing the positions of two vertex operators leaves out a subgroup of infinite volume. Dividing over this infinite volume typically yields a vanishing amplitude. In some cases a finite quantity can be obtained by cancelling the infinite volume of the modular group against the infinte factor $\delta(0)$ arising from the target space zero mode integration [10, 9]. This cancellation is relatively well understood in noncritical string theory but is also believed to happen in other models for example for strings propagating in AdS_3 [11]. As there is no general lore we offer only a few comments on this issue which hopefully clarify the situation to some extent.

While the two infinite factors are in general unrelated, in noncritical string theory dilatations involve translations of the Liouville field because of the background charge. At the technical level one derives two point functions in noncritical string theory by starting with a three point amplitude which is free of divergences and using the ground ring structure that relates it to two point functions [13, 12]. Breaking of target space translation invariance is crucial in this approach because the three point function does not have a momentum conservation delta function. In the example at hand we do not know if there is some algebraic structure similar to the ground ring so we proceed in a somewhat empirical fashion. We first investigate the CFT two point function. The formal expression turns out to be divergent. We investigate the nature of the divergences by regularizing the amplitude and find that the divergences come from contributions of the on-shell states of the undeformed theory describing FZZT branes. We then use the results of [12] for string boundary two point functions for FZZT branes to cancel the aforementioned divergences against the modular group volume. Having sketched the idea we now give the details.

Using (3.24) we find that the formal expression for the CFT two point function has the following contributions

$$
\langle\Phi_{-P_1}(\nu),\Phi_{-P_2}(\nu)\rangle_{\rm CFT} = C_0(P_1,P_2)+C_1(P_1,P_2)+C_1(P_2,P_1)+C_\mu(P_1,P_2)+C_\eta(P_1,P_2)
$$

where

$$C_0(P_1,P_2) = \xi_1\xi_2\langle\Phi^\delta_{|P_1|}e^{-iP_1X_0},\Phi^\delta_{|P_2|}e^{-iP_2X_0}\rangle_{\rm CFT}\,, \tag{8.2}$$

$$C_1(P_1,P_2) = \xi_1\xi_2\int\limits_{-\infty}^{+\infty}d\omega\int\limits_0^\infty dQ\frac{f(Q)d_{-P_2}(\omega)}{(\omega+i\epsilon)^2-Q^2}\langle\Phi^\delta_{|P_1|}e^{-iP_1X_0},\Phi^\delta_{Q}e^{-i\omega X_0}\rangle_{\rm CFT}\,, \tag{8.3}$$

$$C_\mu(P_1,P_2) = \xi_1\xi_2\int\limits_{-\infty}^{+\infty}d\omega_1\int\limits_{-\infty}^{+\infty}d\omega_2\int\limits_0^\infty dQ_1\int\limits_0^\infty dQ_2\frac{f(Q_1)f(Q_2)d_{-P_1}(\omega_1)d_{-P_2}(\omega_2)}{[(\omega_1+i\epsilon_1)^2-Q_1^2][(\omega_2+i\epsilon_2)^2-Q_2^2]}$$
$$\times\langle\Phi^\delta_{Q_1}e^{-i\omega_1X_0},\Phi^\delta_{Q_2}e^{-i\omega_2X_0}\rangle_{\rm CFT}\,, \tag{8.4}$$

$$C_\eta(P_1,P_2) = 4\delta^2\xi_1\xi_2\int\limits_{-\infty}^{+\infty}d\omega_1\int\limits_{-\infty}^{+\infty}d\omega_2\frac{d_{-P_1}(\omega_1)d_{-P_2}(\omega_2)}{(\delta^2+\omega_1^2)(\delta^2+\omega_2^2)}\langle\Phi^\delta_{\vartheta}e^{-i\omega_1X_0},\Phi^\delta_{\vartheta}e^{-i\omega_2X_0}\rangle_{\rm CFT} \tag{8.5}$$

Here for brevity we denoted the normalization factors as $\xi_1 = \xi_0(-P_1)$, $\xi_2 = \xi_0(-P_2)$ (see (4.22)). We also dropped the coordinate dependence of the two point functions: the notation $\langle\mathcal{O}_1,\mathcal{O}_2\rangle_{\rm CFT}$ is used for two point functions on a unit disc with operators $\mathcal{O}_1,\mathcal{O}_2$ inserted at opposite points on the disc boundary.

Except for the last one all of the above four expressions contain divergences. We will see that these divergences arise from integration over the target space zero modes. The first expression can be readily computed using (2.13) and

$$\langle e^{-iX_0\omega_1}(x_1)e^{-iX_0\omega_2}(x_2)\rangle_{\rm CFT} = \delta(\omega_1+\omega_2)|x_1-x_2|^{-2\omega_1^2} \tag{8.6}$$

to obtain

$$C_0(P_1,P_2) = \xi_1\xi_2|C_\delta(P_1)|^2\delta(|P_1|-|P_2|)\delta(P_1+P_2)\,.$$

This expression contains $\delta(0)$ when P_1 and P_2 have opposite signs but this is not the case we are interested in for computing the particle production rate. We assume both $P_{1,2}>0$ and thus can set[7] $C_0(P_1,P_2)=0$.

To exhibit the nature of divergences in $C_1(P_1,P_2)$ and $C_\mu(P_1,P_2)$ we regularize the above expressions by inserting an extra exponent $e^{-i\sigma X_0}$ inside the normal product in each correlator of the form $\langle\Phi^\delta_{Q_1}e^{-i\omega_1X_0},\Phi^\delta_{Q_2}e^{-i\omega_2X_0}\rangle_{\rm CFT}$ so that it is replaced by

$$\langle\Phi^\delta_{Q_1}e^{-i\omega_1X_0},\Phi^\delta_{Q_2}e^{-i(\omega_2+\sigma)X_0}\rangle_{\rm CFT} = |C_\delta(Q_1)|^2\delta(Q_1-Q_2)\delta(\omega_1+\omega_2+\sigma)\,.$$

Substituting this into the above expressions for $C_1(P_1,P_2)$ we obtain

$$C_1^\sigma = \xi_1\xi_2|C_\delta(P_1)|^2f(P_1)\frac{d(-P_1-\sigma)}{(-P_1-\sigma+i\epsilon)^2-P_1^2}\,. \tag{8.7}$$

We observe that in the limit $\sigma\to0$ the divergence comes from restricting on the mass shell the expression $((\omega+i\epsilon)^2-Q^2)^{-1}$. The later can be interpreted as the asymptotic free field propagator. The divergence has the form

$$C_1^{\rm div} = \xi_1\xi_2|C_\delta(P_1)|^2f(P_1)\frac{d(-P_1)}{2P_1\sigma}\,. \tag{8.8}$$

[7] This can be done more mathematically accurately by first regularising this expression, as will be done below, and then removing the regulator.

After the same substitution done for $C_\mu(P_1, P_2)$ we obtain by taking integrals in Q_1, ω_1

$$C_\mu^\sigma = 2\mu_r \xi_1 \xi_2 \int_{-\infty}^{+\infty} d\omega d_{-P_1}(\omega) d_{-P_2}(-\omega - \sigma) I^\sigma(\omega) \tag{8.9}$$

where

$$I^\sigma(\omega) = \int_{-\infty}^{+\infty} \frac{Q^2 dQ}{[(\omega + i\epsilon_1)^2 - Q^2][(\omega + \sigma - i\epsilon_2)^2 - Q^2](\delta^2 + Q^2)} .$$

The integral $I^\sigma(\omega)$ can be computed by taking the residues in the complex upper half plane at $Q = i\delta$, $Q = \omega + i\epsilon_1$, $Q = -\sigma - \omega + i\epsilon_2$. We obtain

$$I^\sigma(\omega) = -\frac{\pi\delta}{(\omega^2 + \delta^2)^2} - \frac{\pi i}{\sigma(\omega^2 + \delta^2)} .$$

Here the second term which is divergent came from the residues taken at $Q = \omega + i\epsilon_1$, $Q = -\sigma - \omega + i\epsilon_2$. Thus again, as in the case of C_2^σ, the divergences come when one of the asymptotic propagators is put on shell. The divergence can be isolated to be

$$C_\mu^{\rm div} = -\frac{2\pi i \mu_r \xi_1 \xi_2}{\sigma} \int_{-\infty}^{+\infty} d\omega \frac{d_{-P_1}(\omega) d_{-P_2}(-\omega)}{\omega^2 + \delta^2} \tag{8.10}$$

The point of doing the above calculations in detail was to show that the divergences arise when one of the expressions of the form $((\omega + i\epsilon)^2 - Q^2)^{-1}$ is reduced to one of the values $\omega = \pm Q = \pm P_{1,2}$. Since such an expression originates as a term in (3.24) that stands at $\Phi_Q^\delta e^{-i\omega X_0}$ we see that the corresponding state is projected onto an on-shell value. Furthermore the propagator itself contains on-shell delta functions $\delta(Q \pm \omega)$ so that we can interpret the divergence at hand as the one arising in a CFT two-point function for two on-shell operators in the undeformed theory: $\langle \Phi_{Q_1}^\delta e^{-iQ_1 X_0}, \Phi_{Q_2}^\delta e^{-iQ_2 X_0} \rangle_{\rm CFT}$. The last expression formally evaluated contains $\delta(0)$.

So far we have considered only the CFT two-point function. For open strings of the FZZT brane in two dimensional string theory a consistent string theory two point function can be extracted from the ground ring relations [12]. Formula (D.12) of [12] taken for $b = 1$ in our notations reads[8]

$$\langle \Phi_{Q_1}^\delta e^{-iQ_1 X_0}, \Phi_{Q_2}^\delta e^{-iQ_2 X_0} \rangle_{\rm str} = 2\pi Q_1 |C_\delta(Q_1)|^2 \delta(Q_1 - Q_2) . \tag{8.11}$$

(It is assumed here that $Q_{1,2} > 0$.) The appearance of the Q_1 factor in the above expression was to be expected from relativistic invariance in the asymptotic region.

The fact that the divergences in a two point function of the time-dependent theory arise from overlaps of pairs of states that are on-shell in the undeformed theory implies

[8]To obtain formula (8.11) from formula (D.12) of [12] one needs to take into account different normalizations given in (2.9) of [12] and the fact that in [12] a Euclidean signature time field is considered. The latter results in an overall sign change.

that we can use formula (8.11) to extract finite results from those divergent pieces. The string theory two point function can be defined as[9]

$$\langle \Phi_{-P_1}(\nu)\Phi_{-P_2}(\nu)\rangle_{\rm str} := 2\pi i \lim_{\sigma\to 0} \sigma \langle \Phi_{-P_1}(\nu), : e^{-i\sigma X_0} \left|\frac{\partial}{\partial X_0}\right| \Phi_{-P_2}(\nu) :\rangle_{\rm CFT} \tag{8.12}$$

where $|\frac{\partial}{\partial X_0}|$ stands for an operator formally defined as

$$: \left|\frac{\partial}{\partial X_0}\right| e^{-i\omega X_0} := |\omega| : e^{-i\omega X_0} : .$$

With this prescription applied to the divergent parts (8.8), (8.10) we readily obtain an expression that can be recognized as the expression in the right hand side of (8.1) multiplied by a factor of 2, that is

$$\langle \Phi_{-P_1}(\nu)\Phi_{-P_2}(\nu)\rangle_{\rm str} = \int_0^\infty dQ[\alpha_{P_1}(Q)\beta_{P_2}(Q) + \alpha_{P_2}(Q)\beta_{P_1}(Q)] = 2\int_0^\infty dQ\alpha_{P_1}(Q)\beta_{P_2}(Q) \tag{8.13}$$

where in the last step we used the second relation in (6.5). Formula (8.13) is the main result of this section. It means that for the class of outgoing particles with wavefunctions (7.9) the pair creation amplitudes computed via Bogolyubov coefficients (6.4) coincide with a suitably defined string theory two-point function.

9. String theory three point function

In this section we will compute the string three point function of the operators $\Phi_{-P}(\nu)$. This amplitude does not contain any divergences and the computation is straightforward. It boils down to computing the CFT three point function which is then stripped of its dependence on the insertion points. Thus in the following we will suppress the insertion points.

We start by substituting expansions (3.6), (3.24) into the three point function

$$\langle \Phi_{-P_1}\Phi_{-P_2}\Phi_{-P_3}\rangle_{\rm CFT} \tag{9.1}$$

and using (2.20), (8.6). This yields the following expression

$$\langle \Phi_{-P_1}\Phi_{-P_2}\Phi_{-P_3}\rangle_{\rm str} = 2\pi \int_{-\infty}^{\infty} dt\, h(t,-P_1)h(t,-P_2)h(t,-P_3) \tag{9.2}$$

where the function $h(t,-P)$ is given in (3.19). The above integral can be conveniently computed by using the Fourier transform of the integrand. The integrand up to an exponential factor $\exp(-it(P_1+P_2+P_3))$ is a polynomial in functions $W(t)$. The Fourier transforms of powers $W^n(t)$ can be readily computed using the differential equation

$$W^2 = \delta W - \partial_t W\,. \tag{9.3}$$

[9]The numerical factor in (8.12) depends on the particular choice of the regularization parameter σ. We fixed this ambiguity essentially by hand to yield the correct final expression. A more satisfactory solution to this problem would involve identifying uniquely a σ-regularization of the volume of residual modular group.

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This differential equation implies the recurrence relation for the Fourier transforms $\widehat{W^n}(\tilde{\omega})$

$$\widehat{W^{n+1}}(\tilde{\omega}) = \delta(1 + \frac{i\tilde{\omega}}{n})\widehat{W^n}(\tilde{\omega}) \tag{9.4}$$

that can be solved as

$$\widehat{W^n}(\tilde{\omega}) = \delta^n \frac{\Gamma(n + i\tilde{\omega})}{\Gamma(1 + i\tilde{\omega})(n-1)!}\hat{W}(\tilde{\omega})\,. \tag{9.5}$$

This implies that the amplitude has the following form

$$\langle \Phi_{-P_1}\Phi_{-P_2}\Phi_{-P_3}\rangle_{\rm str} = \frac{\xi_0(-p_1)\xi_0(-p_2)\xi_0(-p_3)P(p_1,p_2,p_3)}{p_1p_2p_3(p_1+i)(p_2+i)(p_3+i)}\hat{W}(p_1+p_2+p_3) \tag{9.6}$$

where $P(p_1, p_2, p_3)$ is a polynomial. The last one can be computed using (9.5). Using Maple we arrive at the following compact looking result

$$\langle \Phi_{-P_1}\Phi_{-P_2}\Phi_{-P_3}\rangle_{\rm str} = \frac{2\pi}{15}F(p_1)F(p_2)F(p_3)\frac{(\Pi(p_1)+\Pi(p_2)+\Pi(p_3))}{\sinh[\pi(p_1+p_2+p_3)]} \tag{9.7}$$

where

$$F(p_i) = \frac{\nu^{ip_i}}{\pi\sqrt{\mu_r\delta p_i}(1+p_i^2)}\,, \tag{9.8}$$

$$\Pi(p) = 2p + 5p^3 + 3p^5\,. \tag{9.9}$$

We conjecture that this expression, being integrated with wave function factors $\phi_i(p_i)$, $i = 1, 2, 3$, gives a triplet creation amplitude due to string interaction. The outgoing states in this amplitude have wavefunctions of the form (7.9).

10. Choice of the "in" vacuum

In this section we will discuss how one can specify completely a reasonable set of "in" vacua and will find the corresponding expressions in terms of the "out" states. To complete the definition of $|0\rangle_{\rm in}$ we need to specify the initial quantum state of the η mode described by the upside-down harmonic oscillator (7.7) with symplectic form (7.3).

Note that the value of ν can be offset by a suitable time translation so from now on we will set it equal to 1. The canonically conjugated momentum to the quantized coordinate $\hat{\eta}$ is

$$\hat{\pi} = \frac{\tilde{\mu}}{4\delta^2}\widehat{\partial_t\eta}$$

and the Hamiltonian reads

$$\hat{H} = \frac{2\delta^2}{\tilde{\mu}}\hat{\pi}^2 - \frac{\tilde{\mu}}{8}\hat{\eta}^2\,. \tag{10.1}$$

The canonical pair $\hat{q}$ and $\hat{p}$ is related to $\hat{\eta}$, π as

$$\hat{q} = \frac{1}{2}\tilde{\mu}^{1/2}\hat{\eta} - 2\delta\tilde{\mu}^{-1/2}\hat{\pi}\,,$$

$$\hat{p} = \frac{\tilde{\mu}^{1/2}}{4\delta}\hat{\eta} + \tilde{\mu}^{-1/2}\hat{\pi}\,. \tag{10.2}$$

Since the Hamiltonian (10.1) is unbounded from below there is no vacuum state for this system. On the other hand any reasonable initial state for the decaying FZZT brane should have the tachyonic mode localized around the zero value. We thus require that

$$ {}_{\rm in}\langle 0|\hat{\eta}^2|0\rangle_{\rm in} = a\,. \tag{10.3} $$

Since the value of the η variable in the new vacuum is $u_* = 2\delta$ it is reasonable to require $a \ll 4\delta^2$. Among all states satisfying the constraint (10.3) one can find the state for the $\hat{\eta}$, $\hat{\pi}$ system that minimizes the expectation value of the energy (10.1) [22, 21]. The result is a Gaussian wave function

$$ \psi_0(\eta) = (2a\pi)^{-1/4} e^{-\eta^2/4a}\,. \tag{10.4} $$

The time evolution of this wave function is [21]

$$ \begin{aligned} \psi(\eta,t) &= A(t)e^{-\eta^2 B(t)}\,, \\ B(t) &= \frac{\tilde{\mu}}{8\delta}\tan(\phi - i\delta t)\,, & A(t) &= (2\pi)^{-1/4}[b\cos(\phi - i\delta t)]^{-1/2} \\ \phi &= \arctan\left(\frac{2\delta}{\tilde{\mu}a}\right)\,, & b &= (\sin(2\phi))^{-1/2}(4\delta/\tilde{\mu})^{1/4}\,. \end{aligned} \tag{10.5} $$

For any value of a the wave packet rapidly spreads. For small values of a this happens due to the uncertainty principle while for large values due to the unboundedness of the potential. For large times the speed of the spread is exponential, proportional to $e^{\delta t}$. The η degree of freedom however is described by an upside down oscillator only asymptotically, at some point the interaction effects become significant with the energy of the η degree of freedom being lost into radiation.

The initial wave function (10.4) satisfies

$$ \hat{\pi}\psi_0(\eta) = \frac{i}{2a}\eta\psi_0(\eta)\,. \tag{10.6} $$

In terms of the operators $\hat{q}$, $\hat{p}$ this condition reads

$$ \hat{p}\psi_0 = C_a\hat{q}\psi_0\,, \quad C_a = \frac{(1 + i\frac{2\delta}{\tilde{\mu}a})}{2\delta(1 - i\frac{2\delta}{\tilde{\mu}a})}\,. \tag{10.7} $$

We add the condition $\hat{p}|0\rangle_{\rm in} = C_a\hat{q}|0\rangle_{\rm in}$ to the conditions $a^{\rm in}_P|0\rangle_{\rm in} = 0$ characterizing the "in" state. These equations have a unique solution in terms of the "out" oscillators. Using (6.21), (6.12) one finds that a solution to equations $a^{\rm in}_P|0\rangle_{\rm in} = 0$ has the following general form

$$ |C\rangle = F[\hat{q}]\,\hat{G}|0\rangle_{\rm out} \tag{10.8} $$

where

$$ \hat{G} = \exp\left[\frac{1}{2}\int_0^\infty\!\!\int_0^\infty dQ_1 dQ_2\left[\int_0^\infty dP\gamma^*_{Q_1}(P)\beta^*_P(Q_2) + \gamma^*(q_1)(\beta(q_2) - \alpha(q_2))\right]a^{{\rm out}\dagger}_{Q_1}a^{{\rm out}\dagger}_{Q_2}\right]. \tag{10.9} $$

One further finds that

$$ \hat{p}\hat{G}|0\rangle_{\rm out} = i\hat{q}\hat{G}|0\rangle_{\rm out}\,. \tag{10.10} $$

Equation (10.7) fixes the function F so that the state

$$|a\rangle_{\rm in} \equiv \exp\left[\frac{1+iC_a}{2}\hat{q}^2\right]\hat{G}|0\rangle_{\rm out} \tag{10.11}$$

solves the conditions $a_P^{\rm in}|a\rangle_{\rm in} = 0$ and $\hat{p}|a\rangle_{\rm in} = C_a\hat{q}|a\rangle_{\rm in}$. It can be used to compute generic pair creation amplitudes ${}_{\rm out}\langle Q_1Q_2|a\rangle_{\rm in}$. It would be interesting to find out whether there is a prescription for computing such amplitudes via a suitable string theoretic two point function. Using (6.14), (6.21) one finds that

$$\int_0^\infty dP\,\gamma_Q(P)\Phi_{-P}(\nu) + \sqrt{2/\tilde{\mu}}\gamma(q)\delta_- = \Psi_{-Q}(\nu) + \dots \tag{10.12}$$

where the dots stand for out-states of positive frequency. This combination is a natural candidate for a vertex operator whose string theory two point function gives the ${}_{\rm out}\langle Q_1Q_2|a\rangle_{\rm in}$ amplitude. The difficulty in defining a two point function for these operators is in defining it for terms involving $\tilde{\delta}(\nu)$. One would need to make sense of correlators of the form

$$\langle e^{-\delta X_0}(x_1)e^{i\omega X_0}(x_2)\rangle \tag{10.13}$$

where the correlator is taken in the unperturbed theory (in the far past). The naive zero mode integral involved in this expression diverges. One could imagine that a prescription defining the two-point function for exponentially blowing up operators like $\tilde{\delta}(\nu)$ can be found and may involve a parameter a that could be identified with the a present in the definition of $|a\rangle_{\rm in}$. At present we have no idea how this can be done.

11. Summary and further directions

In this paper we computed to leading order in δ string vertex operators for the time-dependent model of [2]. The expressions giving time dependent vertex operators for string states asymptoting to "in" and "out" states are given in (3.20), (3.24), (5.14), (5.15). Several special solutions at zero momentum were identified (see (3.29), (3.34), (6.9)). We defined bases of "in" and "out" scattering states corresponding to scaling operators at the associated UV and IR fixed points (see formulas (4.14), (4.18)). The complete set of Bogolyubov coefficients was obtained in (6.7), (6.8), (6.14) and unitarity relations between them checked.

We further discussed the second quantization of this system and identified a codimension one subspace of out-going wave functions for which the pair creation amplitudes are independent of the ambiguity in defining the initial state of the tachyonic mode. We then showed in section 8 that this set of amplitudes can be obtained in the first quantized framework by computing the appropriate string theoretic two-point functions. As we discussed in section 8 the main difficulty with computing string two point functions in general is in the need to define a ratio of two infinite factors. Our computation proceeded in a somewhat ad hoc manner, utilizing the known results for two point functions in noncritical string theories [10, 12]. It is clear that a deeper understanding of string theory two point

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functions and a general method for their computation is needed. For the same codimension one subspace we computed a string three point functions (9.7) and conjectured that it gives a triplet creation amplitude due to string interaction. A possible test of this conjecture could come from an open string field theory description of the model. One could envision a Das-Jevicki type [23] theory with a fundamental cubic vertex. The triplet creation amplitude would be then analogous to the one computed in [17] for a scalar ϕ^3 theory in an expanding universe.

In the main body of the paper we discussed two special features related to the tachyonic nature of the onset of the time-dependent process at hand. The first feature is the need to deal with solutions exponentially blowing up in the far past while the second, related feature, is the ambiguity in defining the quantum initial state of the system (the absence of Fock vacuum). We suspect that both of these peculiarities generically take place in tachyon decay processes. We discussed the second feature in some detail in section 10 where, following the ideas of [21, 22], we constructed a family of physically reasonable initial states (10.11) in the second quantized oscillator state space.

As for the presence of exponentially blowing up vertex operators in the scattering spectrum consider a string background whose spatial CFT has relevant operators Φ_i. Following [20] one can consider a time-dependent CFT perturbed by a marginal operator of the form $\lambda\Phi_i e^{X_0 p_i}$ where λ is a coupling constant and $p_i > 0$. Such a background will have an infinitesimal deformation corresponding to varying λ. In the far past the corresponding vertex operator vanishes while generically it will not vanish in the far future where it is described by some superposition of outgoing scattering states. Assume that the time-dependent CFT has a conserved inner product (1.17). We then see that the only way to reconcile the existence of vertex operators vanishing in the far past but not in the far future with the conservation of inner product is to admit the existence of solutions blowing up in the far past. A vertex operator asymptoting to positive frequency "out" states will generically blow up in the far past. Thus computing *general* amplitudes will require making sense of correlators involving vertex operators that exhibit such blow up behavior. The standard first quantized formalism of string theory does not provide us with a prescription to compute such correlators. One could hope however that it can be augmented by such a prescription or prescriptions. If such a prescription was found one could use it to reconstruct the corresponding initial state in the secondary quantization, possibly matching it to one of the physical initial states constructed in section 10. We hope to come back to this issue for the model studied in this paper in future work. This problem may test the limits of the first quantized formalism for time-dependent problems. It is worth mentioning in this context a problem of UV divergence in the number of emitted closed string particles radiating from decaying D-branes [24]. It was shown that the problem that is present in the first quantized approach gets cured for decaying D0 branes in two dimensional string theory in the second quantized formalism [25]. The divergence disappears when one includes in consideration the initial wave function of the D0 brane.

For the model considered in this paper it would be also interesting to study one-loop amplitudes and the first order backreaction effects due to radiation of open and closed strings. We leave these questions for future work.

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A. Relations for Bogolyubov coefficients

Checking relations (6.5) boils down to proving the following identity

$$\int_{-\infty}^{+\infty} dq \, \frac{q^3}{\sinh[\pi(q-p_1-i\epsilon)]\sinh[\pi(q-p_2+i\epsilon')]} = \frac{p_1^2(p_1-i)^2 - p_2^2(p_2+i)^2}{2\sinh[\pi(p_2-p_1-i\epsilon)]} \tag{A.1}$$

where p_1 and p_2 are any two real numbers. To compute the integral above we first compute a generating function

$$I_{p_1,p_2}(t) = \int_{-\infty}^{+\infty} dq \, \frac{e^{iqt}}{\sinh[\pi(q-p_1-i\epsilon)]\sinh[\pi(q-p_2+i\epsilon')]} \,. \tag{A.2}$$

It can be evaluated by summing the appropriate residues in the q-complex plane

$$I_{p_1,p_2}(t) = \frac{2i}{\sinh[\pi(p_2-p_1-i\epsilon)]}\left(\frac{e^{itp_2}e^{-t} - e^{itp_1}}{1-e^{-t}}\right). \tag{A.3}$$

The right hand side of identity (A.1) can be evaluated now by computing $i[\partial_t^3 I_{p_1,p_2}(t)]_{t=0}$.

The second pair of relations between Bogolyubov coefficients (6.6) follows from the following identity

$$\text{P.V.}\int_{-\infty}^{+\infty} dp \frac{1}{p(1+p^2)^2\sinh[\pi(p-q_1-i\epsilon)]\sinh[\pi(p-q_2+i\epsilon')]} =$$
$$\frac{1}{2\sinh[\pi(q_2-q_1-i\epsilon)]}\left(\frac{1}{q_2^2(q_2+i)^2} - \frac{1}{q_1^2(q_1-i)^2}\right) + \delta(q_1,q_2) \tag{A.4}$$

where

$$\delta(q_1,q_2) = -\frac{\pi^2}{2}\left(\frac{\cosh(\pi q_1)}{\sinh^2[\pi(q_1+i\epsilon)]\sinh[\pi(q_2-i\epsilon')]} + \frac{\cosh(\pi q_2)}{\sinh[\pi(q_1+i\epsilon)]\sinh^2[\pi(q_2-i\epsilon')]}\right). \tag{A.5}$$

This identity can be obtained by taking the integral via summing over the residues in a complex half plane. Substituting the explicit expressions (6.4) into the left hand side of (6.6) and using (A.4) we obtain the right hand side of (6.6) with

$$d(Q_1,Q_2) = -\frac{4}{\delta}|q_1 q_2|^{3/2}\nu^{i(q_1-q_2)}\left(\frac{q_1-i}{q_1+i}\right)\left(\frac{q_2+i}{q_2-i}\right)\delta(q_1,q_2) \tag{A.6}$$

that can be equivalently written as

$$d(Q_1,Q_2) = \langle \delta(\nu), \tilde{\delta}(\nu)\rangle_{\mathrm{KG}}[\alpha(-q_1)\beta(q_2) - \alpha(q_2)\beta(-q_1)] \,.$$

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